Analytical solutions for the preliminary estimation of long-term rates of groundwater inflow into excavations:
Rectangular excavations

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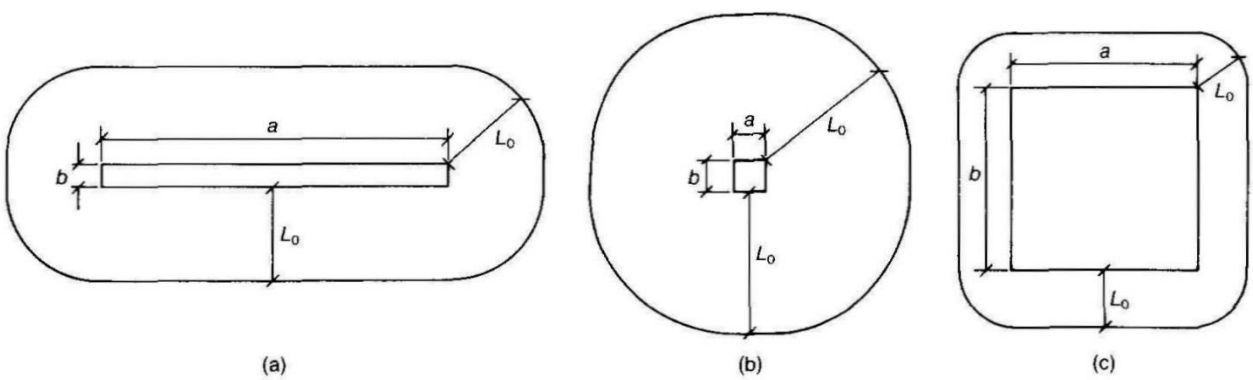
Last update: April 28, 2017

Overview

There are no closed-form analytical solutions for flow into rectangular excavations. Based on the work of Powrie and Preene (1992) approximate approaches for estimating the steady groundwater flow into rectangular excavations in confined aquifers are developed for three cases:

1. Flow into a "long" excavation;
2. Flow into an approximately square excavation with a distant recharge boundary; and
3. Flow into an approximately square excavation with a nearby recharge boundary.

These cases are shown schematically below.



Conceptual models for groundwater flow into rectangular excavations
Reproduced from Powrie and Preene (1992)

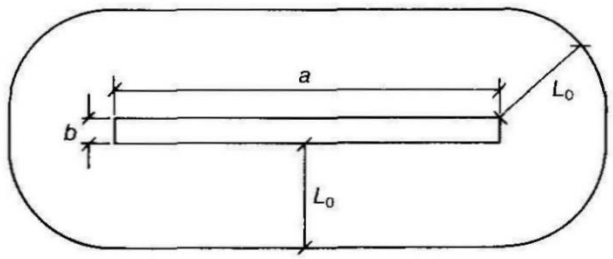
In addition to the simplified analyses, the general "shape factor" approach of Powrie and Preene (1992) is presented and assessed.

Contents

1. Analytical solution for flow into a “long” excavation
2. Analytical solution for flow into an approximately square excavation with a distant recharge boundary
3. Analytical solution for flow into an approximately square excavation with a nearby recharge boundary
4. General analysis of flow into an excavation of any dimensions and distance to the recharge boundary
5. References

1. Flow into a "long" excavation

The conceptual model for a long excavation is shown schematically below (*a* >> *b*).



An approximate expression for the steady inflow to the excavation is developed following an approach presented in Driscoll (1986; Groundwater and Wells, 2nd edition, p. 740). The flow rate into the excavation is approximated as:

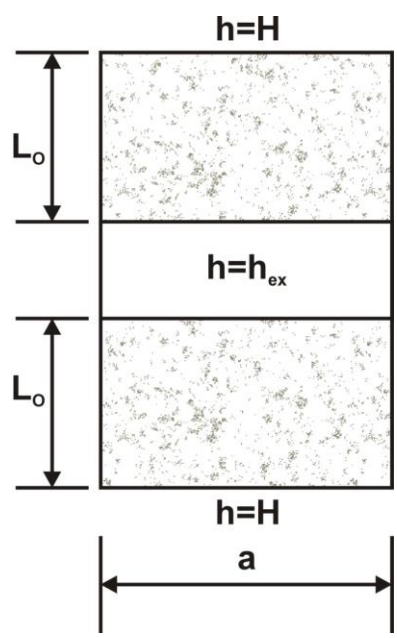
$Q \approx 2 \times$ linear flow into sides + 2 × radial flow into each end semi-circle

1. Linear flow into the long sides of the excavation

The conceptual model for flow into the long side is illustrated below. The inflow into each of the long sides of the excavation is approximated as:

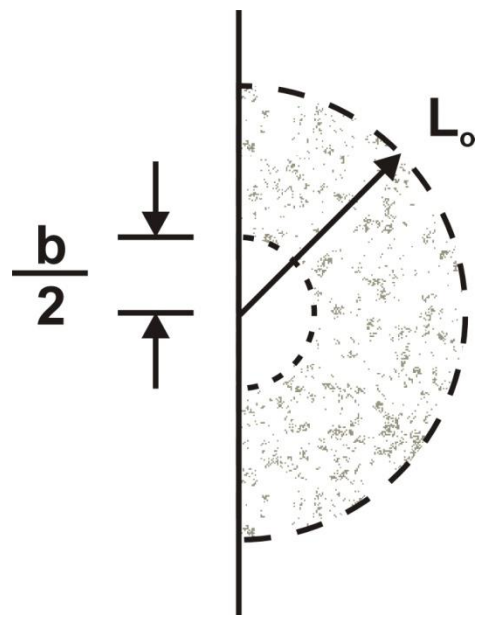
$$Q_S = KD\frac{(H - h_{ex})}{L_o}\ a$$

Here *K* denotes the horizontal hydraulic conductivity and *D* denotes the thickness of the confined aquifer.



2. <u>Radial flow into the ends of the excavation</u>

The conceptual model for the flow into each end of the excavation is shown below.



The inflow into each end of the excavation is approximated from the Thiem solution for steady radial flow:

$$Q_E = \frac{1}{2}\left[2\pi KD\frac{(H-h_{ex})}{\ln\left\{\frac{L_0}{\left(\frac{b}{2}\right)}\right\}}\right]$$

The solution for the flow into each end of the excavation can be written in a simpler form as:

$$Q_E = \pi KD\frac{(H-h_{ex})}{\ln\left\{\frac{2L_0}{b}\right\}}$$

3. <u>Total flow into the excavation</u>

The total flow into the excavation is therefore:

$$Q = 2Q_S + 2Q_E$$

$$= 2\left[KD\frac{(H-h_{ex})}{L_o}a\right] + 2\left[\pi KD\frac{(H-h_{ex})}{\ln\left\{\frac{2L_0}{b}\right\}}\right]$$

Simplifying yields:

$$\boxed{Q = 2KD\,(H-h_{ex})\left[\frac{a}{L_o} + \frac{\pi}{\ln\left\{\frac{2L_o}{b}\right\}}\right]}$$

This solution is identical to Powrie and Preene (1992; Equation [1]). It can also be shown that the solution is the confined flow equivalent to the solution for unconfined flow presented in Driscoll (1986; p. 741).

Powrie and Preene (1992) compared the results of the simplified solution for a long excavation against the results from finite element simulations. Over the range of 0.01 < L_0/a < 1, the results from the solution approximate the numerical results closely (within about ±20%).

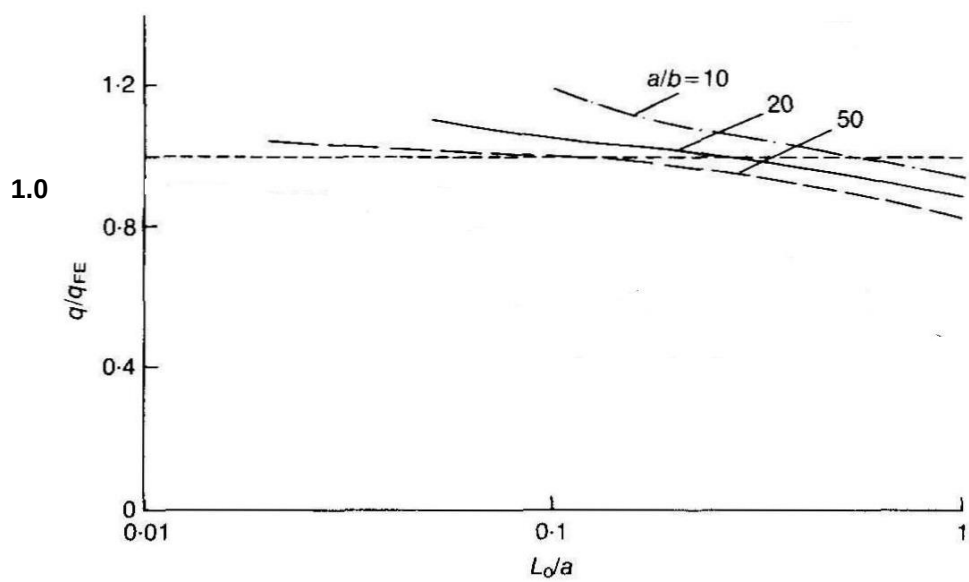
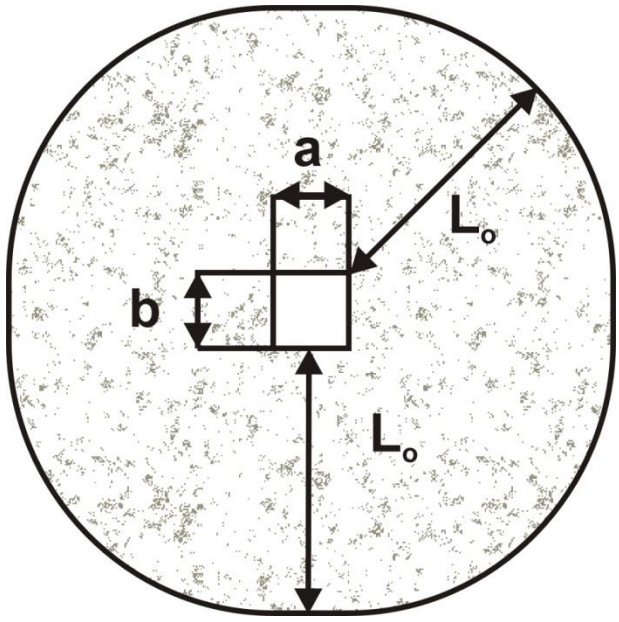


Figure 4. Comparison of results for a long excavation
Reproduced from Powrie and Preene (1992)

2. Flow into an approximately square excavation with a distant recharge boundary

The case of an approximately square excavation with a recharge boundary located at a distance that is much greater than the dimensions of the excavation is shown in illustrated below.



Approximately square excavation (*a* ≅ *b*), distant boundary (*L₀* >> *a*, *b*)

Following the approach of Mansur and Kaufman (1962), the inflow to the excavation is estimated assuming radial flow to an equivalent well:

$$Q = \frac{2\pi KD\ (H - h_{ex})}{\ln\left\{\frac{L_o}{r_{eq}}\right\}}$$

The term r_{eq} represents the equivalent radius of the excavation.

Specification of the equivalent radius of the excavation

Three approaches for specifying the equivalent radius are discussed below. Comparisons made by Powrie and Preene (1992) suggest that neither approach for calculating r_{eq} is universally superior.

Approach #1: Equivalent radius defined on the basis of the equivalent area

For a circular excavation that has the same plan area as the actual rectangular excavation:

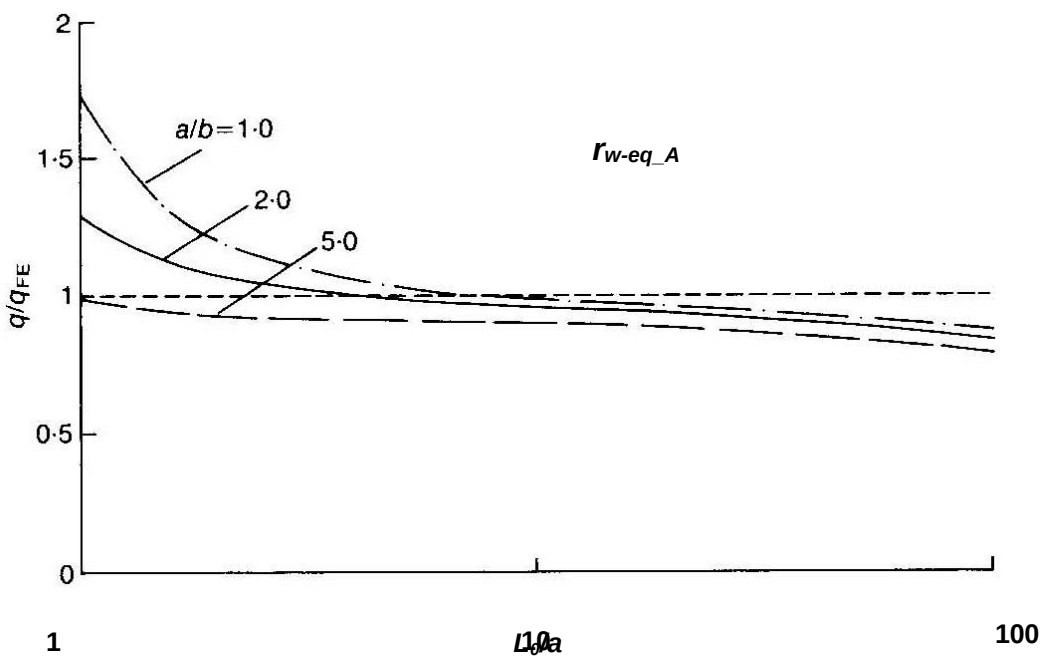
$$\pi r_{w-eq_A}^2 = ab$$

The equivalent radius is therefore:

$$r_{w-eq_A} = \sqrt{\frac{ab}{\pi}}$$

This is Powers and others (2007; Equation [6.8]), and Cashman and Preene (2013; Equation [7.2].

Powrie and Preene`s comparison of the results of finite element simulations and those obtained with the equivalent well solution based on equal area is reproduced below. For values of L_0/a greater than about 2, the results from the simplified radial flow analysis are relatively close the numerical simulations (within ±20%).



Assessment of equivalent single well solution based on area
(Reproduced from Powrie and Preene, 1992; Figure 4)

Approach #2: Equivalent radius defined on the basis of the equivalent perimeter

For an circular excavation that has the same plan perimeter as the actual rectangular excavation:

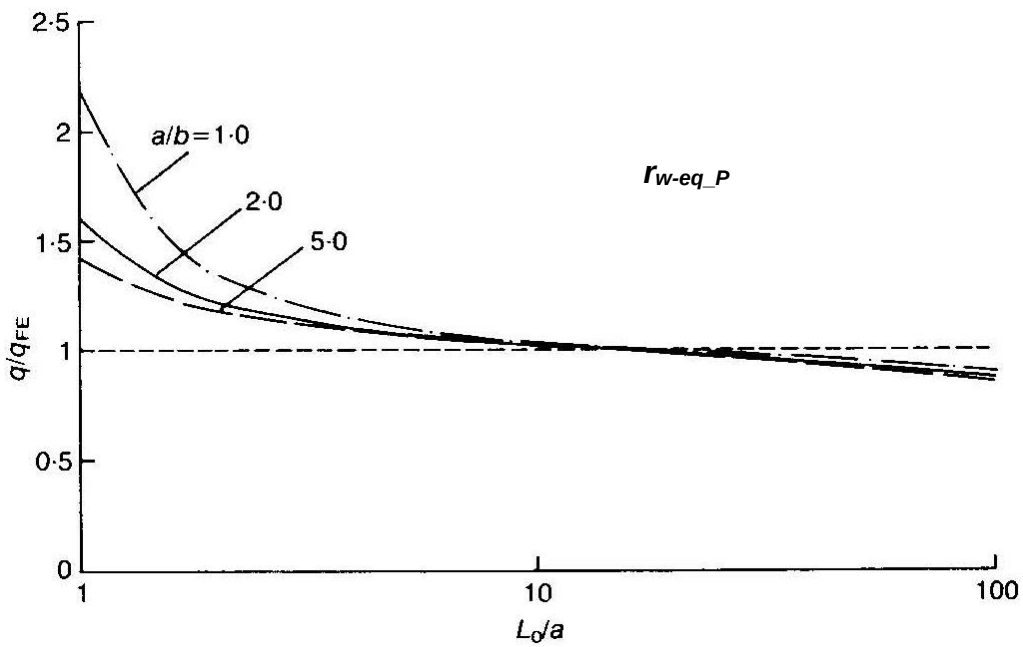
$$2\pi r_{w-eq_P} = 2(a+b)$$

The equivalent radius is therefore:

$$r_{w-eq_P} = \frac{a+b}{\pi}$$

This is Powers and others (2007; Equation [6.9]), and Cashman and Preene (2013; Equation [7.1]).

Powrie and Preene`s comparison of the results of finite element simulations and those obtained with the equivalent well solution based on equal perimeter is reproduced below. For values of L_0/a greater than about 5, the results from the simplified radial flow analysis are relatively close the numerical simulations (within ±10%). The results shown for the equivalent area and equivalent perimeter suggest that neither approach is universally superior.



Assessment of equivalent single well solution based on perimeter

Approach #3: Equivalent radius based on the approach of Mansur and Kaufman (1962)

Mansur and Kaufman (1962) provide the following formula for calculating the equivalent radius of a rectangular excavation (their Equation [39]):

$$r_{w-eq} = \frac{4}{\pi}\sqrt{b_1 b_2}$$

The quantities b_1 and b_2 are the half-length and half-width of the excavation, respectively. Using the present notation:

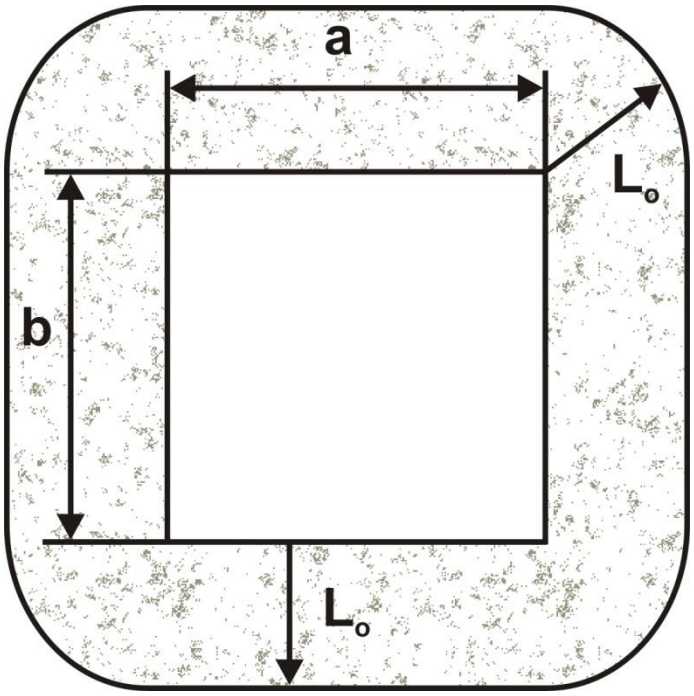
$$r_{w-eq} = \frac{4}{\pi}\sqrt{\frac{a}{2}\frac{b}{2}} = \frac{2}{\pi}\sqrt{ab}$$

We have not been able to find any details on the development of Mansur and Kaufman's formula for the equivalent radius. However, it can be shown that their formula matches the results of the equivalent-area for a square excavation:

$$r_{w-eq_A} = \frac{4}{\pi}\sqrt{\frac{a}{2}\frac{a}{2}} = \frac{2}{\pi}\,a$$

3. Flow into an approximately square excavation with a nearby recharge boundary

The case of a square excavation with a recharge boundary located relatively close to the excavation is shown in illustrated below.

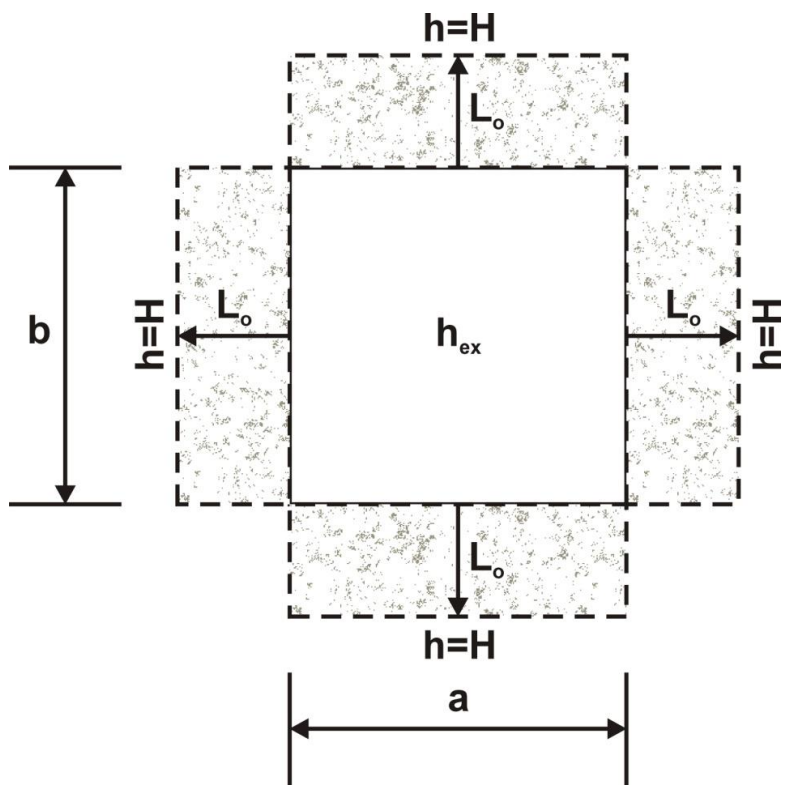


Approximately square excavation (*a* ≅ *b*): nearby boundary (*L_0* ~ *a*, *b*)

For the case of a nearby recharge boundary, Powrie and Preene (1992) again suggest using the equivalent single well approach; however, the solutions are considered to be more approximate than the solution for a distant recharge boundary. Powrie and Preene discuss two approximate approaches for estimating the steady flow into the excavation for this case. The two approximate approaches are developed in the following sections.

Approximate analysis #1

The flow rate is estimated by assuming linear flow into the sides of the excavation, as shown below.



Conceptual model for approximate analysis #1

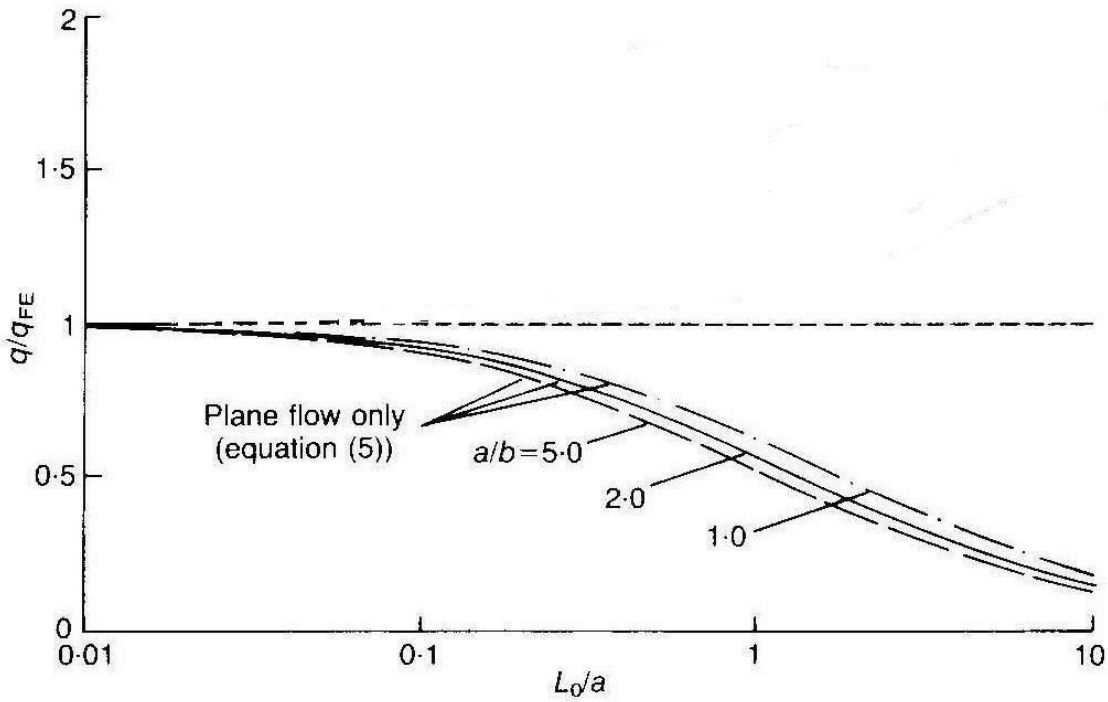
The inflow to the excavation is approximated as the linear flow into each side of the excavation:

$$Q = 2 * \left[KD \frac{(H - h_{ex})}{L_0} \, a \right] + 2 * \left[KD \frac{(H - h_{ex})}{L_0} \, b \right]$$

Collecting terms:

$$\boxed{Q = KD \, (H - h_{ex}) \frac{2(a+b)}{L_0}}$$

The results of the first approximate solution are compared with finite element results below. The results from the first simplified analytical solution are relatively close to the numerical results for values of L_0/a less than 0.1 (within 10%). For values of L_0/a greater than 0.1, the simplified analytical solution underestimates the flows significantly. Powrie and Preene (1992) consider this analysis as *too* approximate, as it neglects flow from the corners.



Assessment of simplified solution #1 for a nearby boundary
(Reproduced from Powrie and Preene, 1992; Figure 5)

Approximate analysis #2

For the second approximate analysis, flow into the corners of the excavation is considered using the average perimeter as defined by Cedergren (1989):

$$\bar{P} = 2\,[a+b] + 4\left[\frac{1}{4}\,2\pi\frac{L_o}{2}\right]$$

Simplifying:

$$\bar{P} = 2(a+b) + \pi L_o$$

Writing:

$$Q = K\frac{(H-h_{ex})}{L_0}A$$

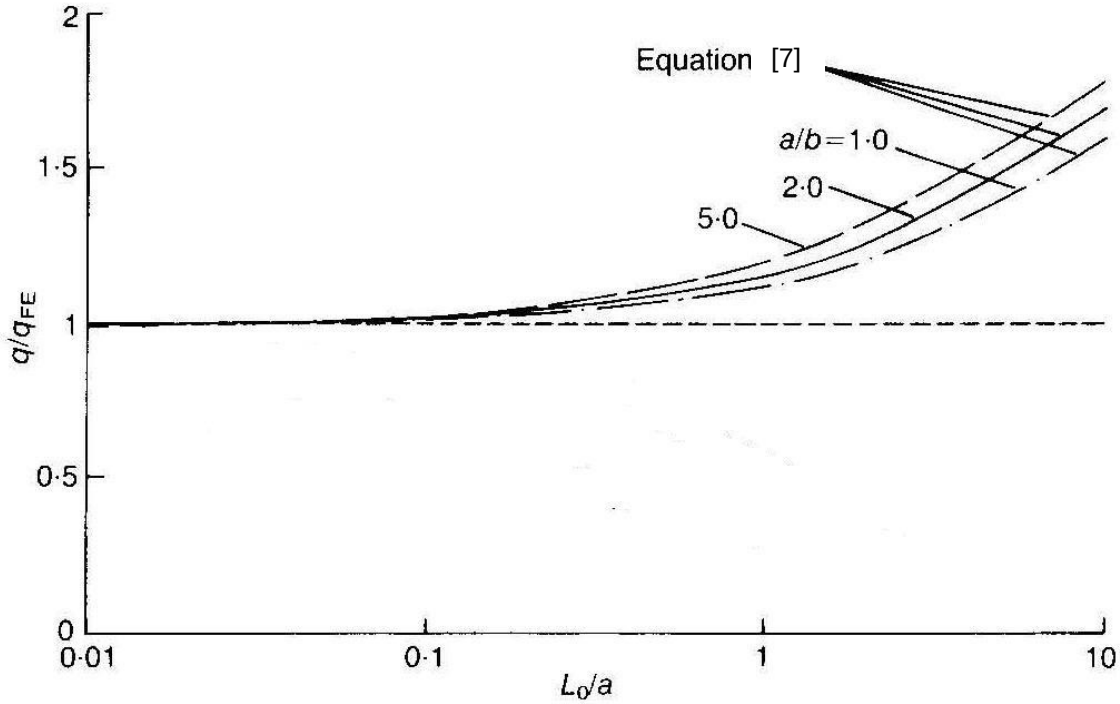
$$= K\frac{(H-h_{ex})}{L_0}\,\bar{P}D$$

$$= KD\,\frac{(H-h_{ex})}{L_0}\cdot[2(a+b)+\pi L_o]$$

Simplifying yields:

$$\boxed{Q = KD(H-h_{ex})\left[\frac{2(a+b)}{L_o}+\pi\right]}$$

The solution for the approximate analysis #2 is identical to Powrie and Preene (1992; Equation 4). The results of the second approximate solution are compared with finite element results in Figure 9. For values of L_0/a less than 0.1, the results match closely. For values of L_0/a greater than 1, the relative errors in the simplified analytical solution increase beyond 20%.



Assessment of the simplified solution #2 for a nearby boundary
(Reproduced from Powrie and Preene, 1992; Figure 5)

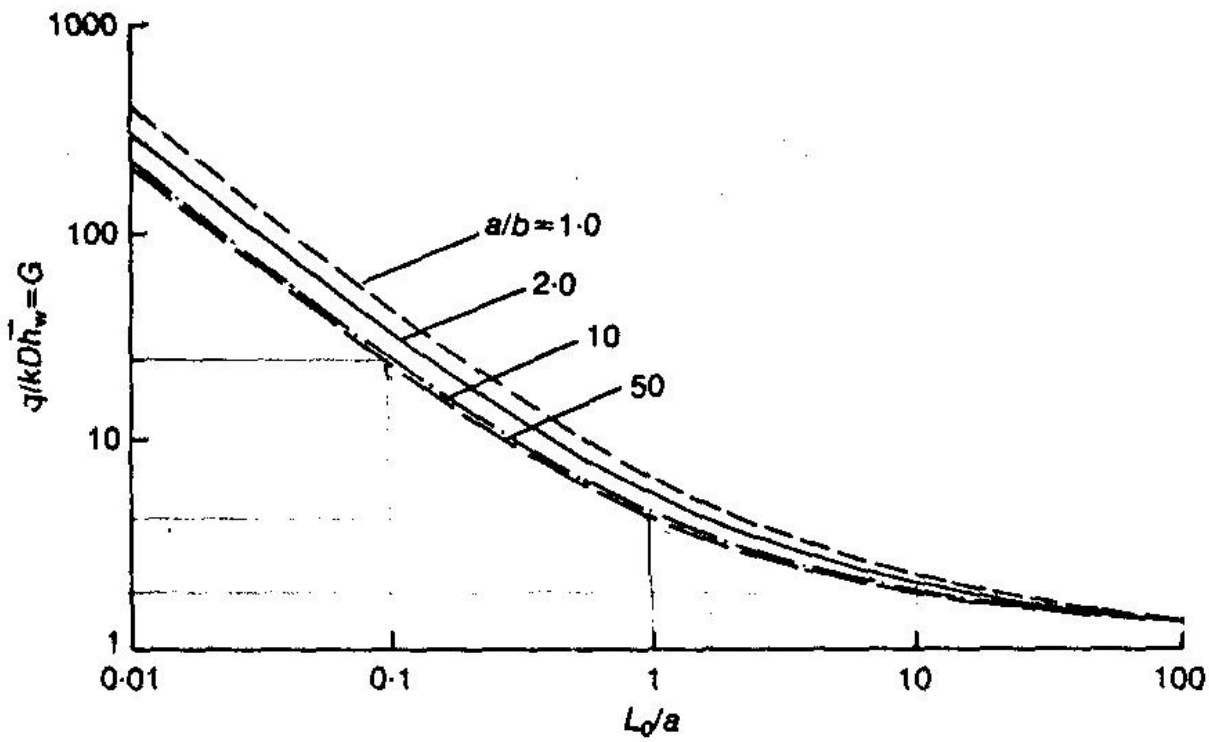
4. Flow into an excavation of any dimensions and distance to the recharge boundary

To estimate the inflow to a rectangular excavation of any dimensions, Powrie and Preene (1992) proposed a general relation of the form:

$$Q = KD\ (H - h_{ex})G$$

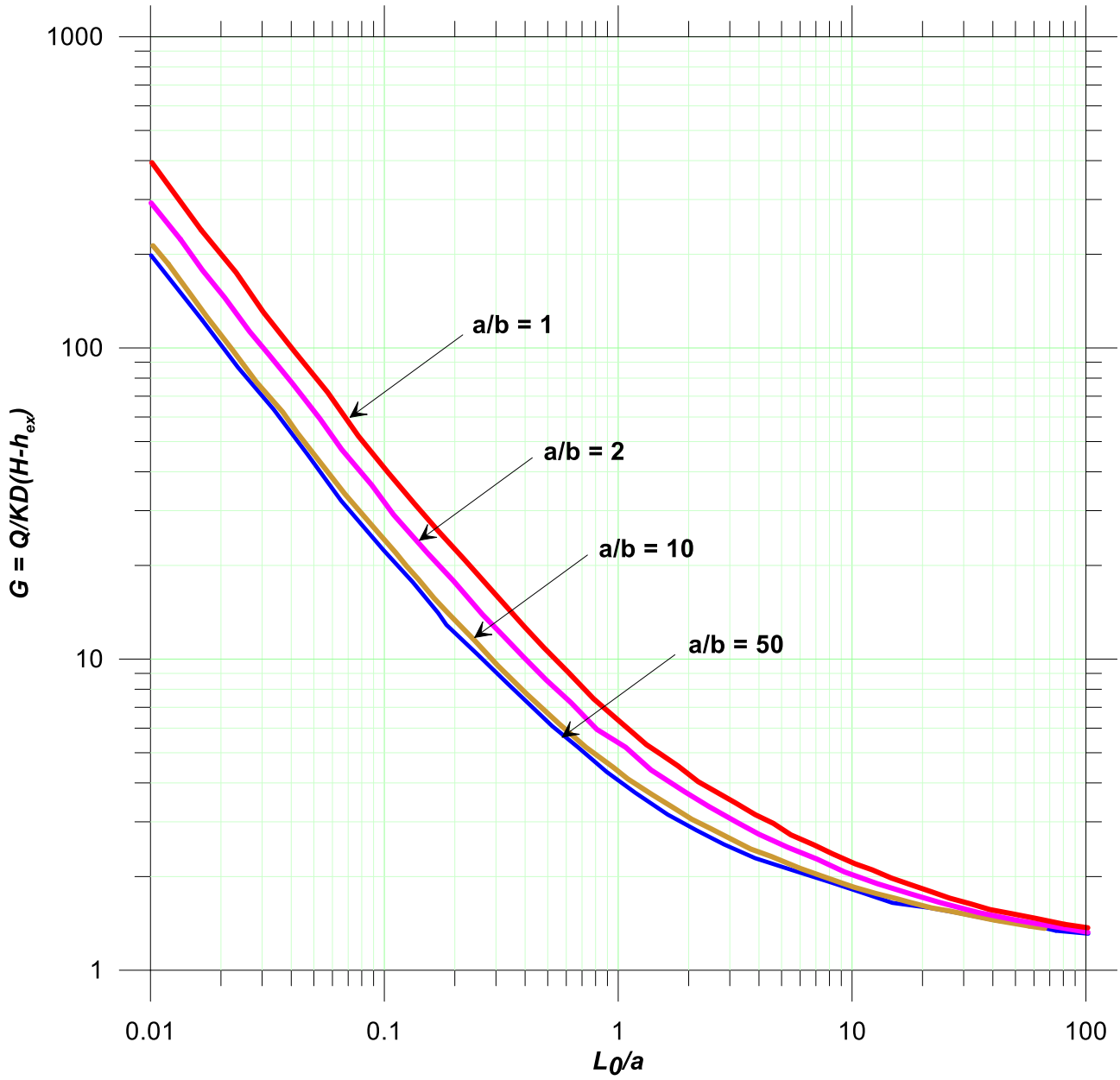
Here *K* is the horizontal hydraulic conductivity, *D* is the aquifer thickness, *H* is the head at the recharge boundary and h_{ex} is the head in the excavation. The parameter *G* is a "shape factor", which may be interpreted as the dimensionless discharge. The shape factor *G* depends on the dimensions $\frac{a}{b}$ and $\frac{L_0}{a}$ (or $\frac{L_0}{b}$).

Powrie and Preene (1992) presented the following plot of *G*, based on the results of a finite-element analysis. The term $\overline{h_w}$ corresponds to the head difference (H-h_{ex}).



Shape factor *G*
(Reproduced from Powrie and Preene, 1992)

To simplify estimation of values of *G*, the curves plotted in Powrie and Preene (1992) have been digitized and re-plotted in Figure 11. For values of *a/b* greater than about 10, the values of *G* approximate closely a single line.



Shape factor *G* re-drafted

Assessment of the Powrie and Preene (1992) shape factor approach

1. The values of Powrie and Preene's shape factor *G* appear to reach asymptotic values for large values of *a/b*. Are the values of *G* for large *a/b* consistent with the simplified results for a "long" excavation?

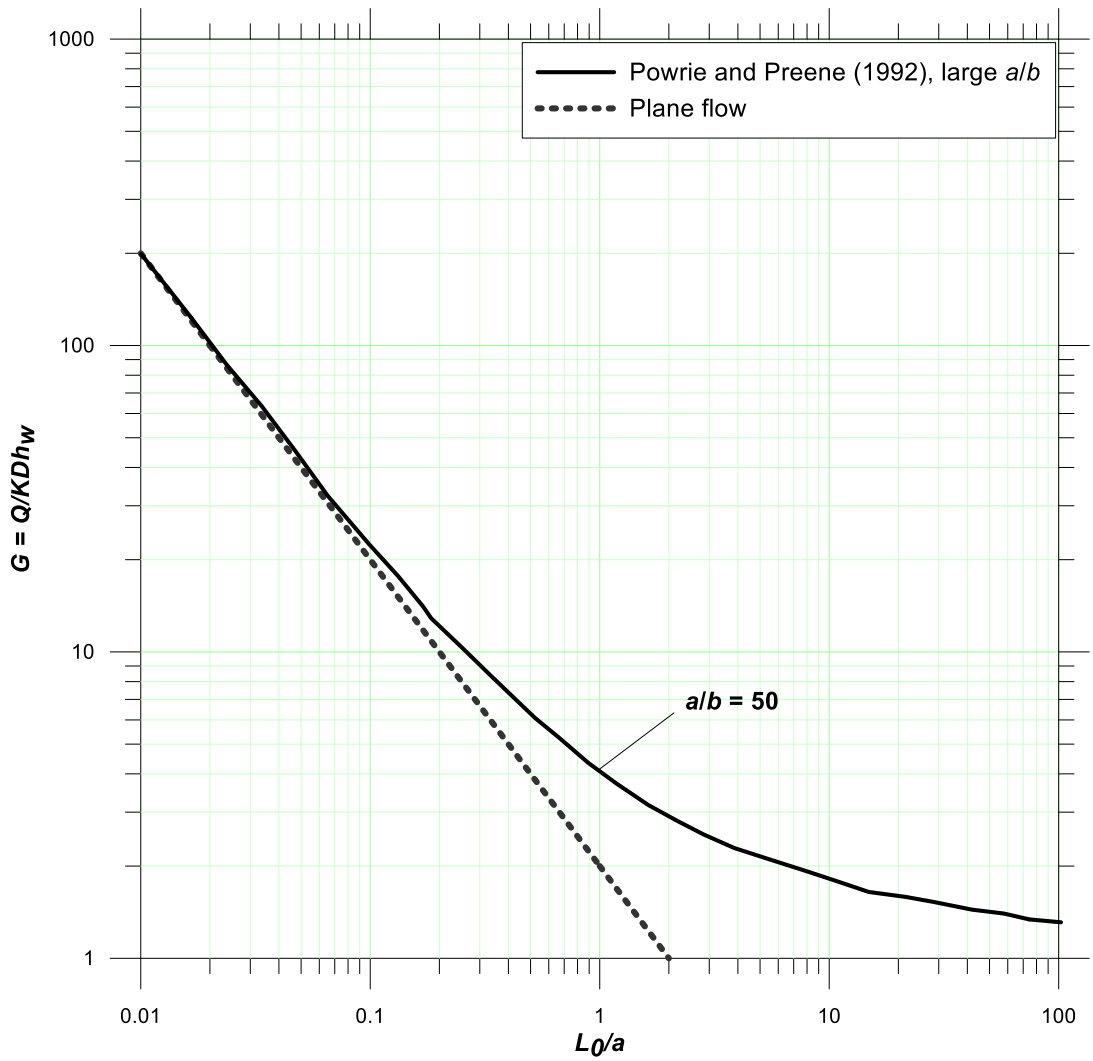
   Recall that for *a*>>*b*, the flow into a "long" excavation is:

$$Q \cong 2KD\ (H - h_{ex})\frac{a}{L_0}$$

   Comparing this equation with the general form yields:

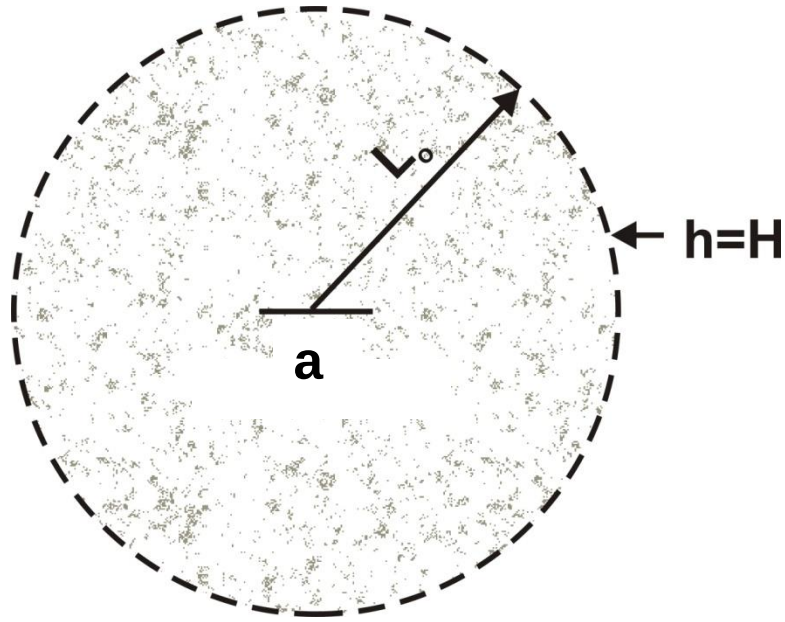
$$G_{long} = 2\frac{a}{L_0} \equiv 2\left(\frac{1}{\left(\frac{L_0}{a}\right)}\right)$$

In the next figure, Powrie and Preene's shape factors for the general case are compared with the simplified plane-flow analysis for a long excavation. As shown in the figure, the shape factor *G* approaches the appropriate asymptotic value for relatively small values of L_0/a.



Assessmennt of the asymptotic case of a "long" excavation

2. The values of the shape factor *G* also appear to converge for large values of L_0/a. Are the values of *G* for large values of L_0/a consistent with the simplified results for a "distant" boundary? As suggested in Figure 5, flow should become progressively more radial than linear as L_0 gets much larger than *a*.

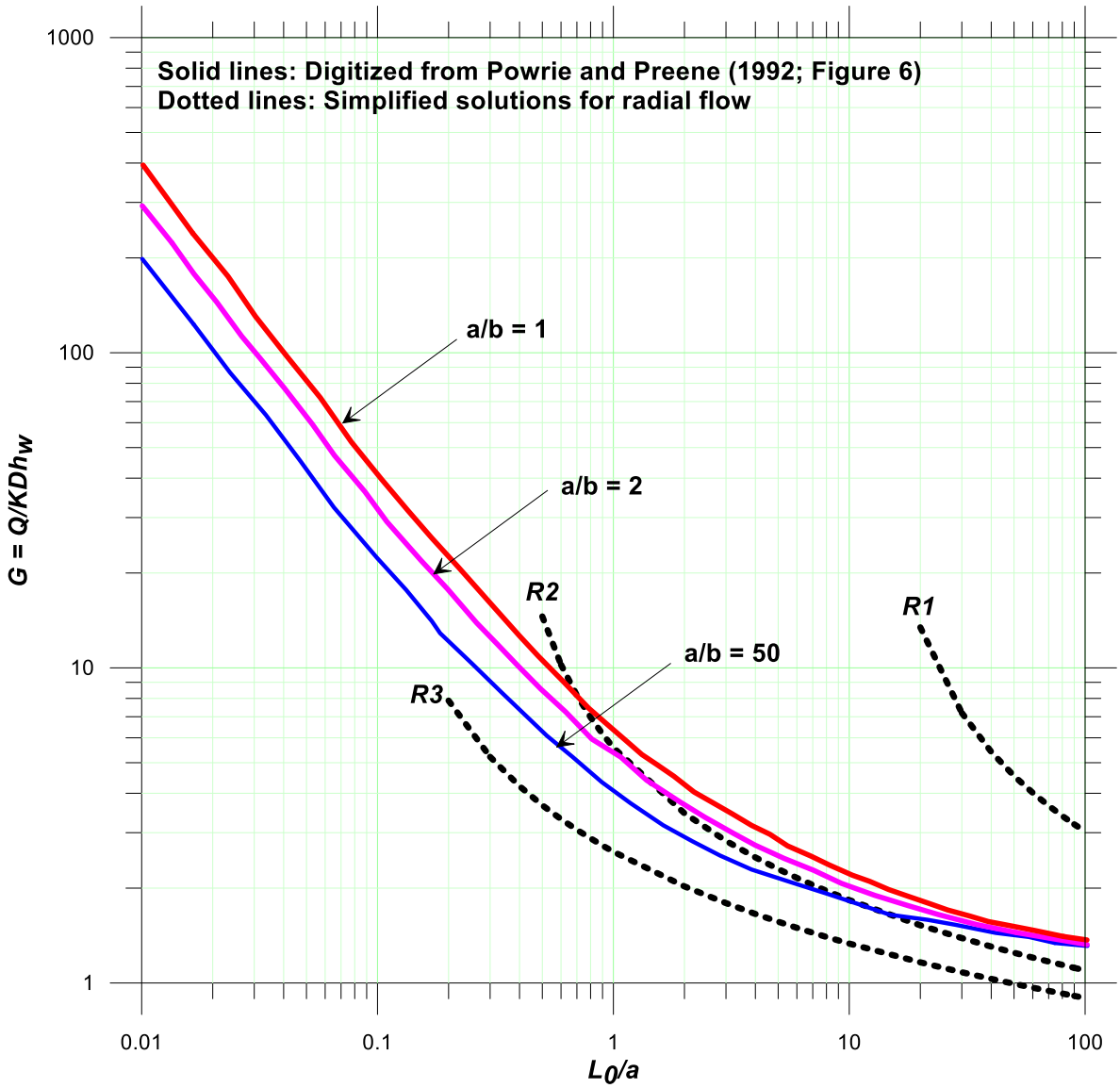


Near-radial flow for a "distant" boundary

The solution for purely radial flow is:

$$Q = 2\pi KD \frac{(H - h_{ex})}{\ln\left\{\frac{L_o}{r_e}\right\}}$$

Powrie and Preene's shape factors for the general case are compared with the simplified radial flow analysis from a distant boundary in the next figure. The radial flow calculations are made for the three different interpretations of the equivalent radius of the excavation.



Assessment of the asymptotic case of a "distant" boundary

The results presented in Figure 14 that are calculated with the value of r_e based on an equal area are designated *R1.* For *a*/*b* = 50:

$$r_{e_A} = \sqrt{\frac{a \cdot \left(\frac{a}{50}\right)}{\pi}} = \frac{a}{12.533} = 0.0798\ a$$

$$\frac{L_0}{r_{e_A}} = 12.533\ \frac{L_0}{a}$$

The results presented in the figure that are calculated with the value of r_e based on an equal perimeter are designated *R2.* For *a*/*b* = 50:

$$r_{e_P} = \frac{a + \left(\frac{a}{50}\right)}{\pi} = 0.325\ a$$

$$\frac{L_0}{r_{e_P}} = 3.080\ \frac{L_0}{a}$$

The results presented in the figure that are calculated with the value of r_e based on the formula of Mansur and Kaufman (1962) are designated *R3.* For *a*/*b* = 50:

$$r_{e_MK} = \frac{2}{\pi}\sqrt{a \cdot \left(\frac{a}{50}\right)} = 0.090\ a$$

$$\frac{L_0}{r_{e_MK}} = 11.107\ \frac{L_0}{a}$$

The results plotted in the figure suggest that the asymptotic trends of Powrie and Preene's shape factors are consistent with the general trends of the calculations with the simplified radial flow analyses. The closest match is obtained with the equivalent radius of the excavation estimated on the basis of the 'equal perimeter' approach.

5. References

Cashman, P.M., and M. Preene, 2013: **Groundwater Lowering in Construction**, 2nd edition, CRC Press, Boca Raton, Florida.

Driscoll, F.G., 1986: **Groundwater and Wells**, 2nd edition, Johnson Division, St. Paul, Minnesota.

Mansur, C.I., and R.I. Kaufman, 1962: Dewatering, Chapter 3 in *Foundation Engineering*, G.A. Leonards (ed.), McGraw-Hill Book Company, New York, New York, pp. 241-350.

Powers, J.P., A.B. Corwin, P.C. Schmall, and W.E. Kaeck, 2007: **Construction Dewatering and Groundwater Control**, 3rd edition, John Wiley & Sons, Inc., Hoboken, New Jersey.

Powrie, W., and M. Preene, 1992: Equivalent well analysis of construction dewatering systems, *Géotechnique*, vol. 42, no. 4, pp. 635-639.