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Response of a Finite-Diameter Well to an Instantaneous Charge of Water[1]

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Abstract. A solution is presented for the change in water level in a well of finite diameter after a known volume of water is suddenly injected or withdrawn. A set of type curves computed from this solution permits a determination of the transmissibility of the aquifer. (Key words: Aquifer tests; groundwater; hydraulics; permeability)

INTRODUCTION

Ferris and Knowles [1954] introduced a method for determining the transmissibility of an aquifer from observations of the water level in a well after a known volume of water is suddenly injected into the well. (See also *Ferris et al.* [1962]). They reasoned that for practical purposes the well may be approximated by an instantaneous line source in the infinite region, for which the residual head differences due to the injection are described by

$$h = (V/4\pi Tt)e^{-r^2S/4Tt} \tag{1}$$

where

- h = change in head at distance r and time t due to the injection;
- r = distance from the line source or center of well;
- t = time since instantaneous injection;
- V = volume of water injected;
- T = transmissibility of aquifer;
- S = coefficient of storage of aquifer.

They reasoned further that the head H in the injected well would be described closely by (1) when r is set equal to the effective radius r_s [*Jacob*, 1947, p. 1049] of the screen or open hole. Then, since r_s is small, the exponential approaches unity quickly, so that the equation approaches $H = V/4\pi Tt$, which can be written

$$T = V(1/t)/4\pi H \tag{2}$$

To the extent that the equation is valid for a well of finite diameter, a determination of the transmissibility can be obtained from the slope of a plot of head H versus the reciprocal of time $(1/t)$.

Since the volume of water injected into the well is $\pi r_c^2 H_0$, where r_c is the radius of the casing in the interval over which the water level fluctuates and H_0 is the initial head increase in the well, equation 1 can be written

$$h/H_0 = (r_c^2/4Tt)e^{-r^2S/4Tt} \tag{3}$$

and equation 2 can be written

$$H/H_0 = r_c^2/4Tt \tag{4}$$

Recently *Bredehoeft et al.* [1966] demonstrated by means of an electrical analog model of a well-aquifer system that equation 3 gives a satisfactory approximation of the head in an injected well only after the time t is large enough for the ratio H/H_0 to be very small (see Figure 1). The observed discrepancy appears to arise from the assumption that the injected well can be approximated by a line source.

We present here an exact solution for the head in and around a well of finite diameter after the well is instantaneously charged with a known volume of water.

ANALYSIS

Consider a nonflowing well cased to the top of a homogeneous isotropic artesian aquifer of uniform thickness, and screened (or open) throughout the thickness of the aquifer (Figure 2). Suppose that the well is instantaneously charged with a volume V of water. (We will consider

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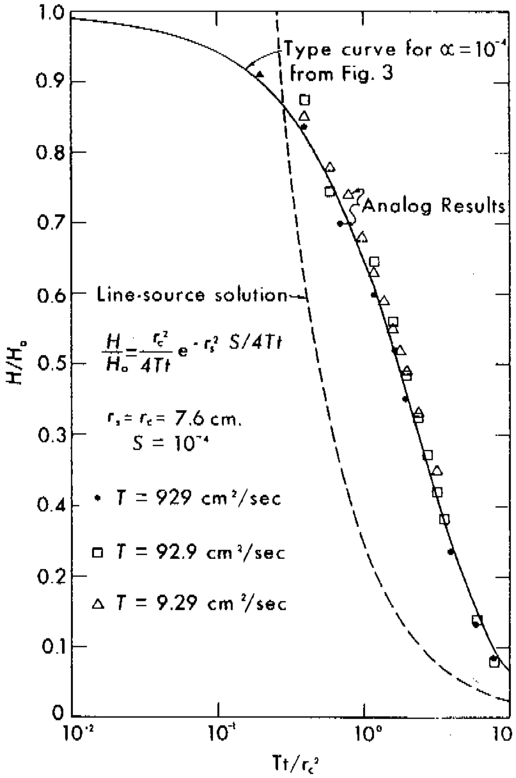

Fig. 1. Comparison of analog results with curve representing line-source solution.

an injection as a positive charge and a withdrawal as a negative one.) The water level in the well instantaneously moves to the height $H_0 = V/\pi r_c^2$ above or below its initial level and immediately begins to return to its initial level according to some function of time $H(t)$. Meanwhile the head in the surrounding aquifer varies according to $h(r, t)$. Our objective is to find a solution for $h(r, t)$ and $H(t)$. The inertia of the column of water in the well will be neglected. (See, in this connection, *Bredehoeft et al.* [1966]). Since the solution to be obtained can be superposed on any initial condition, we can simplify the problem without loss of generality by assuming that the head is initially uniform and constant.

The problem is described mathematically by

$$\partial^2 h/\partial r^2 + 1/r(\partial h/\partial r) = S/T(\partial h/\partial t) \qquad (r > r_s) \qquad (5)$$

$$h(r_s + 0, t) = H(t) \qquad (t > 0) \qquad (5a)$$

$$h(\infty, t) = 0 \qquad (t > 0) \qquad (5b)$$

$$2\pi r_s T[\partial h(r_s + 0, t)]/\partial r = \pi r_c^2(\partial H(t)/\partial t) \qquad (t > 0) \qquad (5c)$$

$$h(r, 0) = 0 \qquad (r > r_s) \qquad (5d)$$

$$H(0) = H_0 = V/\pi r_c^2 \qquad (5e)$$

Equation 5 is the differential equation governing nonsteady radial flow of confined groundwater. (See, for example, *Jacob*, 1950, p. 333.) Boundary condition 5*a* states that after the first instant the head in the aquifer at the face of the well is equal to that in the well. Boundary condition 5*b* states that as r approaches infinity the change in head approaches zero. Equation 5*c* expresses the fact that the rate of flow of water into (or out of) the aquifer is equal to the rate of decrease (or increase) in volume of water within the well. The conditions 5*d* and 5*e* state that initially the change in head is zero everywhere outside the well and equal to H_0 inside the well.

By applying the Laplace transform with respect to time the problem is reduced to

$$\partial^2\bar{h}/\partial r^2 + 1/r\,(\partial\bar{h}/\partial r) = (S/T)\,(p\bar{h}) \qquad (6)$$

$$\bar{h}(\infty, p) = 0 \qquad (6a)$$

$$[\partial\bar{h}(r_s + 0, p)]/\partial r = (r_c^2/2r_sT)[p\bar{h}(r_s + 0, p) - H_0] \qquad (6b)$$

for which the solution is

$$\bar{h}(r, p) = \frac{r_s S H_0 K_0(rq)}{Tq[r_s q K_0(r_s q) + 2\alpha K_1(r_s q)]} \qquad (7)$$

where $q = (pS/T)^{1/2}$, and $\alpha = r_s^2 S/r_c^2$. The solution $h(r, t)$ is the inverse transform, which is available from the analogous problem in heat flow [*Carslaw and Jaeger*, 1959, p. 342]

$$h = \frac{2H_0}{\pi}\int_0^\infty e^{-\beta u^2/\alpha}\{J_0(ur/r_s)\cdot[uY_0(u) - 2\alpha Y_1(u)] - Y_0(ur/r_s)\cdot[uJ_0(u) - 2\alpha J_1(u)]\}\frac{du}{\Delta(u)} \qquad (8)$$

where $\beta = Tt/r_c^2$ and

$$\Delta(u) = [uJ_0(u) - 2\alpha J_1(u)]^2 + [uY_0(u) - 2\alpha Y_1(u)]^2$$

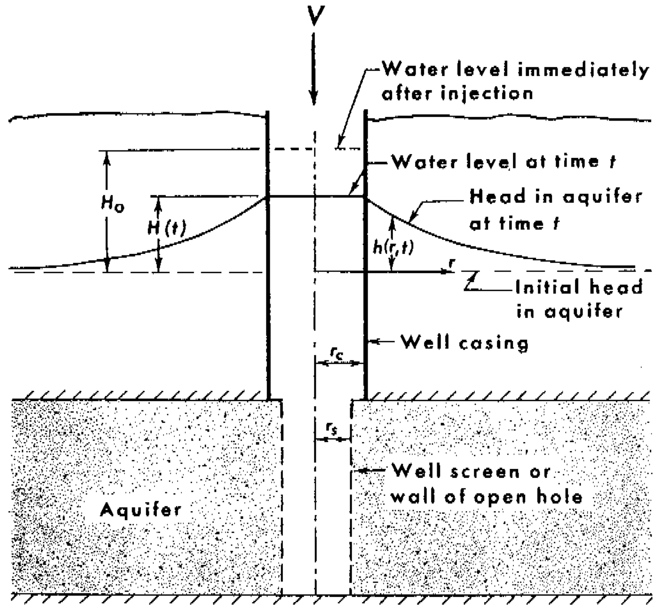

Fig. 2. Idealized representation of a well into which a volume V of water is suddenly injected.

The head $H(t)$ inside the well, obtained by substituting $r = r_s$ in equation 8, is

$$H = (8H_0\alpha/\pi^2) \int_0^\infty e^{-\beta u^2/\alpha}\, du/(u\,\Delta(u)) \quad (9)$$

Values of H/H_0 computed by numerically integrating equation 9 are given in Table 1. Values computed from the line-source solutions, equations 3 and 4, are given in Table 2. In Figure 3 the values from Table 1 are represented as a family of five curves of H/H_0 versus the dimensionless time parameter $\beta = Tt/r_c^2$, one curve for each of five values of the parameter $\alpha = r_s^2S/r_c^2$. Also represented, by a dashed curve, are the values computed from equation 4.

It is apparent from Tables 1 and 2 and from Figure 3 that the line-source solutions 3 and 4 proposed by *Ferris and Knowles* [1954] give a close approximation of the finite-source solution 9 only for large values of the time parameter Tt/r_c^2. The approximation seems to be acceptable for Tt/r_c^2 greater than 100 (or, equivalently, for H/H_0 less than about 0.0025). (In the test at Speedway City, Indiana, used by Ferris and Knowles to exemplify their method, H/H_0 ranged from 0.01 to 0.001, and the value of transmissibility determined from these data agreed fairly well with one obtained by another method.)

A family of type curves plotted on semilogarithmic paper, as in Figure 3, permits a determination of the transmissibility. The method is similar to the Theis graphical method [*Wenzel,* 1942]. A test on a well near Dawsonville, Georgia, will be used to demonstrate the method. This well is cased to 24 m with 15.2-cm (6-inch) casing and drilled as a 15.2-cm open hole to a depth of 122 m. Figure 4 is a reproduction of a chart showing the hydrograph of the well after the sudden withdrawal of a long weighted float from the well. The weight of the float was 10.16 kilograms, and hence by the principle of Archimedes it had displaced a volume of 0.01016 m^3 of water when floating in the well. Its withdrawal was therefore equivalent to a negative charge of $V = 0.01016$ m^3. From the relation $H_0 = V/\pi r_c^2$ the initial head change is found to be $H_0 = 0.560$ m.

The hydrograph in Figure 4 was recorded electrically from a pressure transducer, which was suspended below the water surface in the well. Table 3 lists data from this chart. To determine the aquifer constants the data are

TABLE 1. Values of H/H_0 for a Well of Finite Diameter (computed from equation 9)

	H/H_0				
Tt/r_c^2	$\alpha = 10^{-1}$	$\alpha = 10^{-2}$	$\alpha = 10^{-3}$	$\alpha = 10^{-4}$	$\alpha = 10^{-5}$
1.00×10^{-3}	0.9771	0.9920	0.9969	0.9985	0.9992
2.15×10^{-3}	0.9658	0.9876	0.9949	0.9974	0.9985
4.64×10^{-3}	0.9490	0.9807	0.9914	0.9954	0.9970
1.00×10^{-2}	0.9238	0.9693	0.9853	0.9915	0.9942
2.15×10^{-2}	0.8860	0.9505	0.9744	0.9841	0.9888
4.64×10^{-2}	0.8293	0.9187	0.9545	0.9701	0.9781
1.00×10^{-1}	0.7460	0.8655	0.9183	0.9434	0.9572
2.15×10^{-1}	0.6289	0.7782	0.8538	0.8935	0.9167
4.64×10^{-1}	0.4782	0.6436	0.7436	0.8031	0.8410
1.00×10^{0}	0.3117	0.4598	0.5729	0.6520	0.7080
2.15×10^{0}	0.1665	0.2597	0.3543	0.4364	0.5038
4.64×10^{0}	0.07415	0.1086	0.1554	0.2082	0.2620
7.00×10^{0}	0.04625	0.06204	0.08519	0.1161	0.1521
1.00×10^{1}	0.03065	0.03780	0.04821	0.06355	0.08378
1.40×10^{1}	0.02092	0.02414	0.02844	0.03492	0.04426
2.15×10^{1}	0.01297	0.01414	0.01545	0.01723	0.01999
3.00×10^{1}	0.009070	0.009615	0.01016	0.01083	0.01169
4.64×10^{1}	0.005711	0.005919	0.006111	0.006319	0.006554
7.00×10^{1}	0.003722	0.003809	0.003884	0.003962	0.004046
1.00×10^{2}	0.002577	0.002618	0.002653	0.002688	0.002725
2.15×10^{2}	0.001179	0.001187	0.001194	0.001201	0.001208

plotted on semilogarithmic paper of the same scale as that of the type curves in Figure 3, and this plot is superposed on the type curves. With the arithmetic axes coincident, the data plot is translated horizontally to a position where the data best fit the type curves, as

TABLE 2. Values of H/H_0 for Line-source Approximation of a Well

	H/H_0 from equation 3					
Tt/r_c^2	$\alpha = 10^{-1}$	$\alpha = 10^{-2}$	$\alpha = 10^{-3}$	$\alpha = 10^{-4}$	$\alpha = 10^{-5}$	H/H_0 from eq. 4
1.00×10^{-3}	0.000000	20.52	194.7	243.8	249.4	250.0
2.15×10^{-3}	0.001035	36.35	103.5	115.0	116.2	116.3
4.64×10^{-3}	0.2463	31.44	51.05	53.59	53.85	53.88
1.00×10^{-2}	2.052	19.47	24.38	24.94	24.99	25.00
2.15×10^{-2}	3.635	10.35	11.50	11.62	11.63	11.63
4.64×10^{-2}	3.144	5.105	5.359	5.385	5.388	5.388
1.00×10^{-1}	1.947	2.438	2.494	2.499	2.500	2.500
2.15×10^{-1}	1.035	1.150	1.162	1.163		1.163
4.64×10^{-1}	0.5105	0.5359	0.5385	0.5388		0.5388
1.00×10^{0}	0.2438	0.2494	0.2499	0.2500		0.2500
2.15×10^{0}	0.1150	0.1162	0.1163			0.1163
4.64×10^{0}	0.05359	0.05385	0.05388			0.05388
7.00×10^{0}	0.03558	0.03570	0.03571			0.03571
1.00×10^{1}	0.02494	0.02499	0.02500			0.02500
1.40×10^{1}	0.01783	0.01786				0.01786
2.15×10^{1}	0.01162	0.01163				0.01163
3.00×10^{1}	0.008326	0.008333				0.008333
4.64×10^{1}	0.005385	0.005388				0.005388
7.00×10^{1}	0.003570	0.003571				0.003571
1.00×10^{2}	0.002499	0.002500				0.002500
2.15×10^{2}	0.001163					0.001163

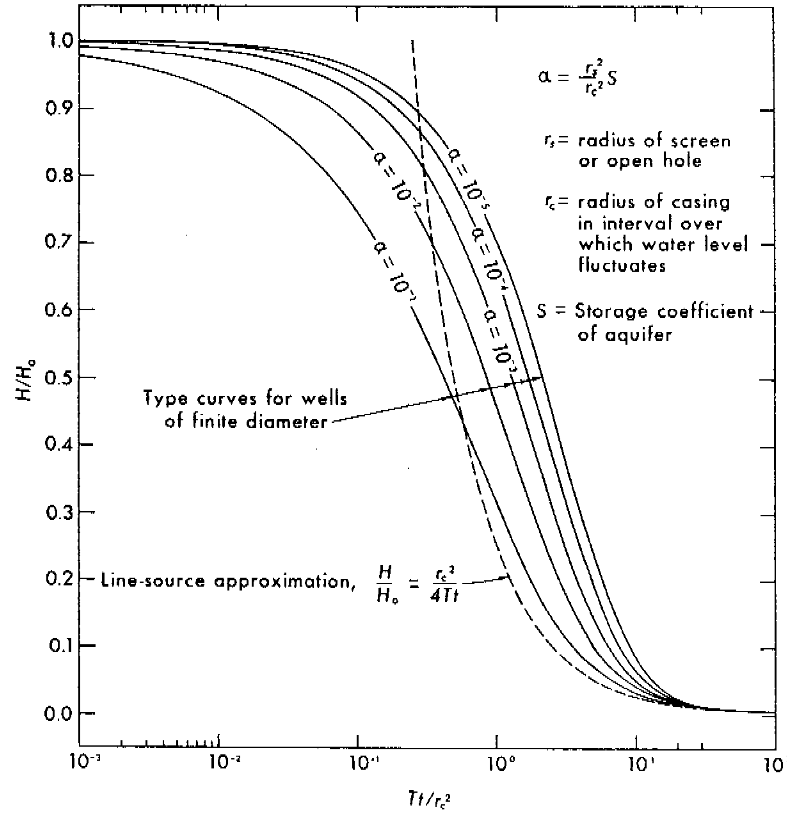

Fig. 3. Type curves for instantaneous charge in well of finite diameter.

shown in Figure 5. In this position the time $t = 11$ sec on the data coordinates is found to overlie the value $Tt/r_c^2 = 1.0$ on the type-curve coordinates. Hence the transmissibility is computed to be

$$T = \frac{1.0 r_c^2}{t} = \frac{(1.0)(7.6)^2}{(11)} = 5.3 \text{ cm}^2/\text{sec}$$

In principle the coefficient of storage can be determined by interpolating from its values for the curves that lie on either side of the data plot in the matched position. Thus, in the example just described, the coefficient of storage would be $S = 10^{-3}$, since for this well $r_c = r_s$, so that $\alpha = S$, and the points fall on the curve for $\alpha = 10^{-3}$. However, because the matching of data plot to the type curves depends upon the shapes of the type curves, which differ only slightly when α differs by an order of magnitude, a determination of S by this method has questionable reliability.

The determination of T is not so sensitive to the choice of the curves to be matched. Whereas the determined value of S will change by an

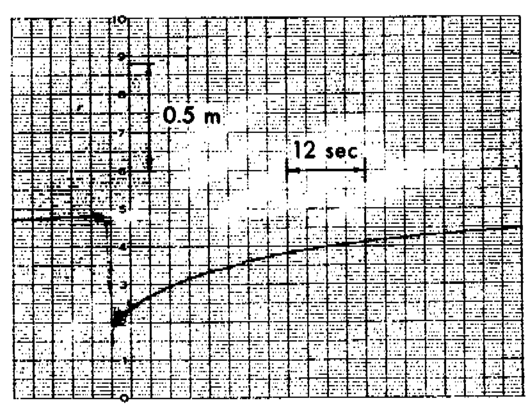

Fig. 4. Hydrograph of well at Dawsonville, Georgia, showing response of water level to the sudden withdrawal of a weighted float.

TABLE 3. Rise of Water Level in Dawsonville Well after Instantaneous Withdrawal of Weighted Float

t (sec)	$1/t$	Head (m)	H (m)	H/H_0
−1		0.896		
0		0.336	0.560	1.000
3	0.333	0.439	0.457	0.816
6	0.167	0.504	0.392	0.700
9	0.111	0.551	0.345	0.616
12	0.0833	0.588	0.308	0.550
15	0.0667	0.616	0.280	0.500
18	0.0556	0.644	0.252	0.450
21	0.0476	0.672	0.224	0.400
24	0.0417	0.691	0.205	0.366
27	0.0370	0.709	0.187	0.334
30	0.0333	0.728	0.168	0.300
33	0.0303	0.747	0.149	0.266
36	0.0278	0.756	0.140	0.250
39	0.0256	0.765	0.131	0.234
42	0.0238	0.784	0.112	0.200
45	0.0222	0.788	0.108	0.193
48	0.0208	0.803	0.093	0.166
51	0.0196	0.807	0.089	0.159
54	0.0185	0.814	0.082	0.146
57	0.0175	0.821	0.075	0.134
60	0.0167	0.825	0.071	0.127
63	0.0159	0.831	0.065	0.116

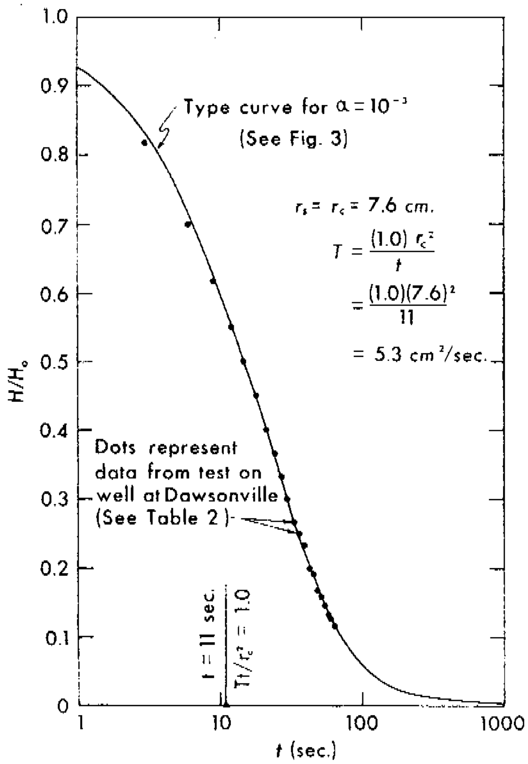

Fig. 5. Plot of data from test at Dawsonville, Georgia, superposed on type curve.

order of magnitude when the data plot is moved from one type curve to another, that of T will change much less. From a knowledge of the geologic conditions and other considerations one can ordinarily estimate S within an order of magnitude and thereby eliminate some of the doubt as to what value of α is to be used for matching the data plot.

Figure 6 shows the data from the test on the Dawsonville well plotted according to the Ferris-Knowles method. The points do not fall along a straight line as postulated in this method but, instead, fall along the trace of the type curve for $\alpha = 10^{-3}$, which has been transferred from Figure 5. Also shown is a straight line through the origin whose slope, when used according to the Ferris-Knowles method, will yield the transmissibility of 5.3 cm²/sec obtained by matching the data to the type curves.

CONCLUSION

The judgment of an experienced hydrologist is needed to decide the significance, if any, of a determination of T by the method of instantaneous charge. As *Ferris et al.* [1962] properly warned

> the duration of a 'slug' test is very short, hence the estimated transmissibility determined from the test will be representative only of the water-bearing material close to the well. Serious errors will be introduced unless the . . . well is fully developed and completely penetrates the aquifer.

Few wells completely penetrate an aquifer, but it is nevertheless possible under some circumstances for a hydrologist to derive useful information from a test on a partially penetrating well. Since the vertical permeabilities of most stratified aquifers are only small fractions of the horizontal permeabilities, the induced flow within the small radius of the cone that develops during the short period of observation is likely to be essentially 2-dimensional. Therefore, the determined value of T would represent approximately the transmissibility of that part

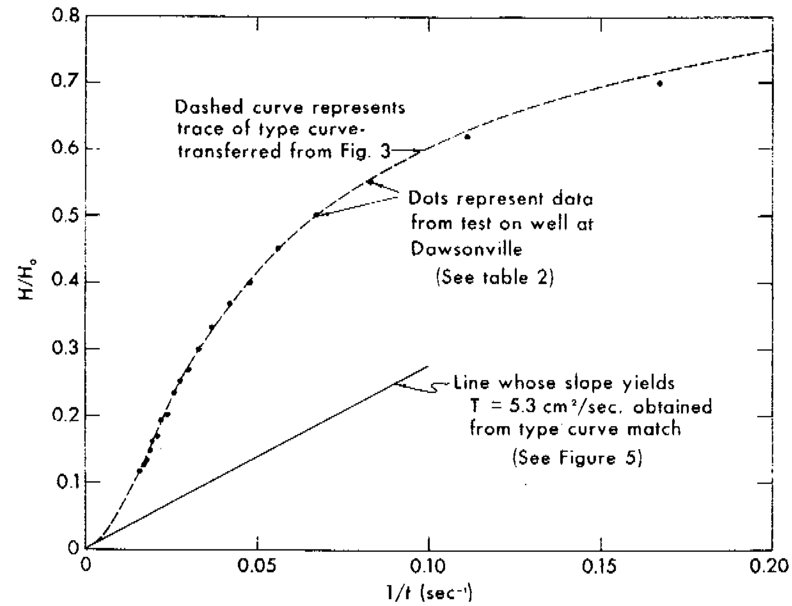

Fig. 6. Data from test on well of Dawsonville, Georgia, plotted according to the Ferris-Knowles method.

of the aquifer in which the well is screened or open, provided that the aquifer is reasonably homogeneous and isotropic in planes parallel to the bedding and provided that the effective radius r_s can be estimated closely.

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