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Paper presented at the
First Joint IAH-CNC/CGS Groundwater Specialty Conference
Montreal, Quebec
October 15-18, 2000

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INTERPRETATION OF CONSTANT-HEAD TESTS: RIGOROUS AND APPROXIMATE ANALYSES

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ABSTRACT The constant-head test is an essential technique of geotechnical and hydrogeological site characterization. The test is used frequently to estimate the hydraulic conductivity of relatively low permeability formations (sparsely fractured rock and clayey soils). The interpretation of these tests has traditionally been based on approximate analyses. The approximate analyses neglect storage in the formation, assume that the well penetrates the formation completely, and assume uniform material properties in the vicinity of the well. Novakowski (1993) developed a rigorous analysis for constant-head tests, considering explicitly storage in the formation, partial penetration, and the presence of a finite-thickness skin. The rigorous analysis involves a mixed boundary condition at the wellbore and cannot be solved by conventional integral transform approaches. Novakowski used a novel application of the Dirac-delta function to derive the Laplace-transform solution. In this study we check the predictions of the Dirac-delta solution against the results of high-resolution finite element analyses. We show that for cases of partial penetration, Novakowski's solution yields erroneous results. We present a correct version of the rigorous solution for Novakowski's problem. Finally, we use the correct, rigorous solution to examine the estimation of hydraulic conductivity from constant-head tests in partially penetrating wells. Our results show that significant errors can be made in interpreting tests when partial penetration is neglected.

1. Introduction

The single-well test involving injection at a constant head is an essential technique of geotechnical and hydrogeological site characterization. The test was developed initially by Lugeon (1933) to assess grouting requirements for dams founded on rock. It is still used for that purpose, but it has also been adopted by geotechnical engineers and hydrogeologists to estimate the hydraulic conductivity of fractured rock and relatively low permeability clayey soils (Shapiro and Hsieh, 1998); and Novakowski et al., 1999; Tavenas et al., 1990). In the geotechnical literature, the test is generally referred to as the Lugeon or packer test; in the hydrogeology literature the test is generally designated the packer or constant-head test.

Conventional interpretations of constant-head tests are based on approximate analyses. Geotechnical applications typically neglect storage in the formation a-priori from the analyses. Hydrogeologic interpretations sometimes consider transient effects, using the analysis of Jacob and Lohman (1952), but more frequently the interpretations also assume steady flow. Zero-storage analyses oblige the interpreter to assume an arbitrary radius of influence for each test. The radius of influence may have little physical meaning in an extensive formation. Furthermore, in the cases of carefully instrumented tests, conventional analyses require neglecting much of the transient data that are collected.

In many cases, the tested interval represents only a relatively small portion of the entire thickness of the formation. In these cases, conventional interpretations of constant-head tests idealize the formation as a confined aquifer of thickness equal to the length of the packed-off interval. The analysis of test data using solutions developed for fully penetrating wells can lead to erroneous results. Drilling and installation of wells frequently results in the development of a zone of altered properties around the wellbore, referred to as a "skin". Conventional analyses assume that the formation has uniform properties. Since the conventional analyses neglect the possibility that a skin exists around the well, they may yield parameter estimates of formation properties that are not representative of the formation.

Few systematic attempts have been made to quantify the errors associated with conventional interpretations of constant-head tests. This lack of attention may be due to the fact that more realistic conceptual models have defied the development of analytical solutions. In 1993, K.S. Novakowski made an important attempt to rectify this situation by developing an analytical solution for a physically-based boundary value problem that incorporates storage in the formation, partial penetration of the well, and the possibility of an altered zone around the well.

In this study, we revisit Novakowski's solution for the case of a partially penetrating well. We show that for cases of partial penetration, Novakowski's solution yields erroneous results. This conclusion is supported by an examination of the properties of the infinite series appearing in Novakowski's solution, and a re-consideration of the boundary conditions implicit in Novakowski's methodology. We present a correct form of the solution. The close match with the results of high-resolution finite-element analyses demonstrates the correctness of our new solution. We use the correct, rigorous solution to evaluate the appropriateness of the conventional methods of analysis. In particular, we try to answer the following question: "How much of an error do we make in our estimation of hydraulic conductivity when we adopt a simplified model to interpret the data from a constant-head test?".

2. Conventional interpretations of constant-head tests

A schematic set-up for a constant-head test is shown on Figure 1, along with an illustration of idealized data obtained during a single-stage test. The principle of the test is simple: the hydraulic head in a packed-off interval is raised suddenly and held constant. The injection rate required to maintain the head rise is monitored through time.

The interpretation of constant-head test data is described in the Earth Manual of the U.S. Bureau of Reclamation (1974) and Ziegler (1976). The horizontal hydraulic conductivity K_H is estimated from the general formula for steady-state flow:

$$K_H = \frac{Q}{F\Delta H} \tag{1}$$

where Q and ΔH denote the flow rate and the applied change in hydraulic head, respectively. The term F is generally referred to as the shape factor. The conventional interpretation assumes purely radial flow within the interval of the packers. For this conceptual model, the flow rate is given by the Thiem solution:

$$Q = \frac{K_H L}{2\pi} \frac{\Delta H}{\ln\left\{\frac{R}{r_w}\right\}} \tag{2}$$

where L is the length of the packed-off interval, r_w is the radius of the wellbore, and R is the radius of influence. Comparing (1) and (2), the shape factor is:

$$F = \frac{L}{2\pi} \frac{1}{\ln\left\{\frac{R}{r_w}\right\}} \tag{3}$$

We refer to this method of interpretation as the "conventional approach", as it appears frequently in the project reports of research organizations that provide important examples of practice, for example, Canada's National Water Research Institute and the United States Geological Survey.

The conventional approach requires specification of a radius of influence, R. In practice, a constant value for R is assumed (Shapiro and Hsieh, 1998; Novakowski et al., 1999). Since R appears in the log term, the interpreted hydraulic conductivity is not particularly sensitive to its value. Nevertheless, the fact that a radius of influence must be assumed, even in an extensive formation far from any physical boundaries, demonstrates the weak theoretical foundation of the conventional approach.

In cases where the tested interval represents a relatively small portion of the entire thickness of the formation, conventional interpretations of constant-head tests idealize the formation as a confined aquifer of thickness equal to the length of each packed-off interval. This approach ignores vertical gradients and and may lead to errors that are difficult to quantify. In an attempt to generalize the conventional approach, several alternative formulae have been presented in the literature. These formulae differ in the shape factor used in Eq. (1). For example, the U.S. Bureau of Reclamation (1974) presents a shape factor originally suggested by Hvorslev (1951). The particular shape factor drawn from the work of Hvorslev is based on the idealized geometry of an ellipsoid embedded in an infinite porous medium. This model is considered more representative of the geometry of a packed-off interval that is only a small portion of the full thickness of a permeable formation.

It is important to note that because all of these shape factor approaches share the same conceptual model of zero-storage in the formation, there is little to distinguish between them. All models that ignore storage in the formation are based on an artificial conception of the hydraulics of a constant-head test, regardless of the "exactness" of the representation of flow around the wellbore.

Conventional approaches for interpreting constant-head tests presume that steady-state conditions prevail. In formations where flow is dominated by a few fractures, this may be a realistic approach since the contribution from storage may be negligible. However, in compressible formations such as soft clays, or in densely fractured rock masses with relatively permeable matrix blocks, the observed flow rate may continue to decline over the duration of a test. Under these circumstances, the conventional approach will require ignoring much of the data collected in a well-instrumented test, and choosing an arbitrary pseudo-steady flow rate.

Since the conventional approach for interpreting constant-head tests has a weak theoretical foundation, is not well-suited for interpreting data from partially penetrating tests, and may require discarding data, there is motivation for developing a rigorous analysis of the constant-head test.

3. Rigorous analysis of the constant-head test

The conceptual model for a rigorous analysis of a constant-head test is shown on Figure 2. The boundary value problem (BVP) considers a well of finite radius that partially penetrates an aquifer of uniform thickness B. The aquifer is confined above and below by impermeable strata. A skin zone of finite thickness, with hydraulic properties different from the formation surrounds the wellbore.

The governing equations for three-dimensional, transient flow in the skin and the formation are:

$$K_{r1}\left(\frac{\partial^2 s_1}{\partial r^2} + \frac{1}{r}\frac{\partial s_1}{\partial r}\right) + K_{z1}\frac{\partial^2 s_1}{\partial z^2} = S_{s1}\frac{\partial s_1}{\partial t} \tag{4}$$

within the skin zone, $r_w \le r_s$ and

$$K_{r2}\left(\frac{\partial^2 s_2}{\partial r^2} + \frac{1}{r}\frac{\partial s_2}{\partial r}\right) + K_{z2}\frac{\partial^2 s_2}{\partial z^2} = S_{s2}\frac{\partial s_2}{\partial t} \tag{5}$$

within the formation, $r_s \le r_w < \infty$

where s_i, K_i, and S_{si} are the head change, hydraulic conductivity, and specific storage of the skin ($i = 1$) and formation ($i = 2$).

At the start of the test, the head change in the skin and the formation is zero, and the head change in the wellbore, s_0, is applied instantaneously. Therefore, the initial conditions are written as:

$$s_1(r,z,0) = s_2(r,z,0) = 0 \qquad \text{(6a)}$$

$$s_w(0^+) = s_{w0} \qquad \text{(6b)}$$

In the general case of an interval open between elevations z_1 and z_2, the inner boundary conditions along the wellbore are written as:

$$\frac{\partial s_1}{\partial r}(r_w,z,t) = 0 \qquad ;\ 0 \le z \le z_1 \qquad \text{(7a)}$$

$$s_1(r_w,z,t) = s_w(t) \qquad ;\ z_1 \le z \le z_2 \qquad \text{(7b)}$$

$$\frac{\partial s_1}{\partial r}(r_w,z,t) = 0 \qquad ;\ z_2 \le z \le B \qquad \text{(7c)}$$

where s_w is the head change in the wellbore itself.

The formation is assumed to be areally extensive and the outer boundary condition is written as:

$$s_2(\infty,z,t) = 0 \qquad \text{(8)}$$

The boundary conditions along the top and bottom of the aquifer are:

$$\frac{\partial s_1}{\partial z}(r,0,t) = \frac{\partial s_2}{\partial z}(r,0,t) = 0 \qquad \text{(9a)}$$

$$\frac{\partial s_1}{\partial z}(r,B,t) = \frac{\partial s_2}{\partial z}(r,B,t) = 0 \qquad \text{(9b)}$$

The BVP is completed with compatibility conditions at the interface between the skin and the formation:

$$s_1(r_s,z,t) = s_2(r_s,z,0) \qquad \text{(10)}$$

$$K_{r1}\frac{\partial s_1}{\partial r}(r_s,z,t) = K_{r2}\frac{\partial s_2}{\partial r}(r_s,z,t) \qquad \text{(11)}$$

As pointed out by Ruud and Kabala (1997), the inner boundary condition for the rigorous problem mixes Type I (specified head) and Type II (no-flow) conditions along the wellbore, over the portions of the well that are screened and cased, respectively. This mixing of conditions precludes an exact solution by conventional integral transform methods.

Novakowski (1993) developed an analytical solution for the BVP using a novel application of the Dirac-delta function in conjunction with the Laplace transform. Novakowski used the delta function to derive a Green's function for the problem, and then integrated the building block solution over the length of the open interval. The final solution was obtained by numerical inversion of the Laplace-transform solution.

Novakowski's Laplace-transform solution is given by:

$$\begin{aligned}\overline{Q}_{Dp}(p) = &-\frac{(\gamma\xi)^{1/2}}{L_D p^{1/2}} \\ &\times \frac{[I_1(q_3)\beta_3 + K_1(q_3)\beta_4]}{\psi}(B_2 - B_1) \\ &- \frac{2L_D}{\pi^2}\sum_{n=1}^{\infty}\frac{q_1[I_1(q_1)\beta_1 + K_1(q_1)\beta_2]}{p\psi} \\ &\times \frac{1}{n^2(B_2 - B_1)}[\sin(\omega B_2) - \sin(\omega B_1)]^2\end{aligned} \qquad \text{(12)}$$

Readers are referred to Novakowski (1993) for definitions of the notation. Note that (12) is identical to Novakowski's Eq. [13], with the exception of the term (pψ) appearing in the summation. The omission of this term from Novakowski's Eq. [13 is a typographical error, as the correct form of the summation appears in Novakowski's Eq. [15].

Novakowski presented no verification of his solution, apart from showing that it collapsed to the solution of Jacob and Lohman (1952) for a fully penetrating well with no skin zone.

4. Problems with the Novakowski solution

In order to check the Novakowski (1993) solution for the case of partial penetration, we compare results from Novakowski's solution with the results of high-resolution numerical simulations using a finite element model (FEM). The FEM code was designed specifically for the analysis of complex aquifer tests (Sudicky, MacQuarrie and Neville, 1990). Novakowski presents selected results in the form of dimensionless type curves. Figure 3 shows the results of Novakowski's solution and the FEM simulations. The results show that while the solutions match closely for the case of full penetration, they diverge as the degree of penetration decreases. The disagreement has been confirmed by repeating the FEM simulations for increasingly finer spatial discretizations.

Inspection of Novakowski's equation [13] reveals that it does not converge. For a partially penetrating well with no skin (*i=1=2*) Eq. (12) reduces to:

$$\begin{aligned}\overline{Q}_D(p) = &\frac{K_1(q_4)}{L_d q_4 K_0(q_4)}(B_2 - B_1) \\ &+ \frac{2L_D}{\pi^2}\sum_{n=1}^{\infty}\frac{q_5 K_1(q_5)}{pK_0(q_5)}\ \frac{1}{n^2(B_2 - B_1)} \\ &\times [\sin(\omega B_2) - \sin(\omega B_1)]^2\end{aligned} \qquad \text{(13)}$$

Evaluating the late-time limiting form of equation (13) yields:

$$\lim_{p \to 0} p\overline{Q}_D(p) = \frac{2}{\pi(B_2 - B_1)} \times \sum_{n=1}^{\infty} \frac{1}{n}\left[\sin(\omega B_2) - \sin(\omega B_1)\right]^2 \tag{14}$$

The series in the last expression is divergent, indicating that equation (12) is the solution to an ill-posed problem.

Independent confirmations of problems with the Novakowski (1993) solution are provided by Cassiani and Kabala (1998) and Cassiani et al. (1999). Cassiani and co-workers use Novakowski's approach to develop an alternate solution for a semi-infinite aquifer. They demonstrate that solutions developed using Novakowski's methodology are influenced by Gibbs effects, and produce total fluxes that are infinite.

The fundamental problem with Novakowski's solution for a partially penetrating well lies with the use of the Dirac-delta function to implement the boundary condition along the wellbore. With Novakowski's approach, the Dirac-delta function is used to describe the change in hydraulic head in the formation due to a point source at the well screen. The delta function is then integrated along the open portion of the well to obtain the change in head in the formation due to a line source. Use of the delta function implies that the change in the hydraulic head is equal to a specified value along the open portion of the well, and is zero elsewhere along the wellbore. The delta function approach does not implement the boundary conditions as specified in Equations (7a-c); instead, the forcing of zero head change along the closed section of the wellbore creates an infinite sink, which gives rise to the infinite total flux observed by Cassiani et al. (1999).

5. Development of an improved analytical solution

We have revisited the BVP for a rigorous model of the constant-head tests and developed a new solution following the approach of Dougherty and Babu (1984). The solution for the dimensionless hydraulic head in the skin and the formation is obtained by using the Laplace transform with respect to dimensionless time, t_D, and the finite Fourier transform with respect to the dimensionless elevation, z_D. The solutions in the Laplace domain are derived by application of the inverse finite Fourier transform. Final results are obtained by numerical inversion of the Laplace-transform solutions using the algorithm of Talbot (1979).

In addition to the governing equations (4-5) and (10-11), the BVP is completed with a statement of mass conservation for the well:

$$Q_w(t) = 2\pi\, r_w K_{r1} \int_{z_1}^{z_2} \frac{\partial s_1(r_w, z, t)}{\partial r} dz \tag{15}$$

To determine the flow rate into the well, we use Darcy's law at the well face in the following dimensionless form:

$$Q_D(p) = -\int_{z_{D1}}^{z_{D2}} \frac{\partial s_{D1}}{\partial r_D}\Big|_{r_D=1} dz_D \tag{16}$$

The solution in the Laplace domain for the flow from the well is:

$$\overline{Q}_D(p) = \frac{1}{p}\left[\frac{\rho}{\Delta'} A_1' I_o(q_1') + A_2' K_0(q_1')\right] + \frac{2}{L_D b_D}\sum_{n=1}^{\infty} \frac{\left[A_1 I_o(q_1) + A_2 K_0(q_1)\right]}{\Delta} \frac{1}{\beta_n} \times \left[\sin(\beta_n z_{D2}) - \sin(\beta_n z_{D1})\right]^2 \tag{17}$$

where the variables are defined in the notation. Complete details of the derivation of the solution, and a FORTRAN code implementing it are available upon request.

Figure 4 shows the results of our revised solution and the previous FEM simulations. The results of our solution agree much more closely with the numerical results. There are still some differences for the smallest degree of penetration considered; however, these differences tend to be exaggerated by the logarithmic axes. The discrepancy may be due to an approximation in the representation of the boundary condition along the wellbore that only becomes significant as the relative length of the open interval becomes relatively small.

6. Interpretation of data from tests in partially penetrating wells

In order to gauge the magnitude of the errors that may arise when applying the conventional method of interpretation, we consider an idealized example. For the example, we consider a confined formation that is 10 m thick and has a uniform horizontal hydraulic conductivity of 10^{-6} m/sec and a specific storage of 10^{-6} m^{-1}. These values are typical for intact clays. The open interval along the wellbore extends from the middle of the formation. The applied head change at the wellbore is 10.0 m.

For the example we use the correct rigorous solution to generate "perfect" discharge vs. time records. We consider an open interval that ranges from 10% to 100% of the formation thickness, and a vertical hydraulic conductivity ranging over three orders of magnitude, from K_z/K_r = 1.0 to 0.001. A typical set of results is shown on Figure 5; results are shown for the case of isotropic hydraulic conductivity, for a range of penetration ratios ρ (=*L*/*B*), from 1.0 (full penetration) down to 0.01. All of the curves shown on Figure 5 are generated with the same specific storage for the formation. The results illustrate a general result: the discharge curves become progressively flatter as the degree of penetration decreases. Rather than providing "proof" of the validity of the zero-storage conceptual model, this response illustrates the difficulties in interpreting aquifer test data when multiple hydraulic processes interact.

Each discharge-time record is analyzed using the conventional approach. We assume a radius of influence of 10 m (the same value as used in Novakowski et al., 1999), and use as a proxy for the steady flow the discharge rate after 1000 seconds. We note that the selection of 1000 seconds is arbitrary, and that the discharge rate has not necessarily stabilized at this time. The results of the conventional analyses are summarized on Figure 6. The results for this example demonstrate that significant errors can be made in the estimation of the horizontal hydraulic conductivity when the conventional approach is applied. The errors are most significant as the degree of penetration decreases, and as the vertical hydraulic conductivity approaches the magnitude of the horizontal conductivity. For highly anisotropic formations (K_z/K_r < 0.01, say), the error arising from the use of the conventional method of interpretation is relatively insensitive to the degree of penetration.

7. Conclusions

Conventional approaches for interpreting the results of constant-head tests are based on a conceptual model of aquifer response that has a weak theoretical foundation. The inability of these approaches to consider storage in the formation and partial penetration of the test interval yield errors that are difficult to quantify. We have reviewed the rigorous solution of Novakowski, and shown that it provides incorrect results for cases of partial penetration. Our results confirm the recent suggestion of Cassiani et al. (1999) that Novakowski's solution is not well founded, and that use of the Dirac-delta function solution methodology is incorrect. We have derived a correct version of the rigorous problem considered by Novakowski. The good agreement between our solution and the results of high-resolution FEM analyses demonstrates that our solution is correct for the case of partial penetration. We use the correct rigorous solution to examine the errors that can arise from the application of a conventional interpretation approach, in the context of a synthetic example. The results from the example demonstrate that significant errors can be made in the estimation of the horizontal hydraulic conductivity when the conventional approach is applied to data from partially penetrating wells.

8. Notation

A_1	$\lambda K_0(q_2 r_{D1})K_1(q_1 r_{D1}) - K_0(q_1 r_{D1})K_1(q_2 r_{D1})$
A_1'	$\lambda' K_0(q_2' r_{D1})K_1(q_1' r_{D1}) - K_0(q_1' r_{D1})K_1(q_2' r_{D1})$
A_2	$\lambda I_1(q_1 r_{D1})K_0(q_2 r_{D1}) + I_0(q_1 r_{D1})K_1(q_2 r_{D1})$
A_2'	$\lambda' I_1(q_1' r_{D1})K_0(q_2' r_{D1}) + I_0(q_1' r_{D1})K_1(q_2' r_{D1})$
b	screen length (z_2-z_1) [L]
b_D	dimensionless screen length (b/z)
B	aquifer thickness [L]
B_D	dimensionless aquifer thickness (B/r_w)
H_0	head change at the wellbore [L]
K_{ri}	radial hydraulic conductivity in region i [LT^{-1}]
K_{zi}	vertical hydraulic conductivity in region i [LT^{-1}]
n	finite Fourier transform variable ($n=0,1,2\ldots\infty$)
p	Laplace transform variable
q_1	$\sqrt{(\alpha\gamma p+\phi_1\beta_m^2)}$
q_1'	$\sqrt{(\alpha\gamma p)}$
q_2	$\sqrt{(p+\phi_2\beta_m^2)}$
q_2'	$\sqrt{p}$
Q_w	volumetric flow rate into or out of the well bore [L^3T^{-1}]
Q_D	dimensionless volumetric flow rate into or out of the wellbore ($Q_w/2\pi(z_2-z_1)K_{r1}H_0$)
r	radial distance [L]
r_D	dimensionless radius (r/r_w)
r_w	well radius [L]
r_s	radial co-ordinate of the outer boundary of the skin zone [L]
r_{Ds}	dimensionless radial co-ordinate of the outer boundary of the skin zone (r_s/r)
s_i	hydraulic head change in region i [L]
s_{Di}	dimensionless hydraulic head change in region i (s_i/H_0)
s_w	hydraulic head change in the wellbore [L]
s_{Dw}	dimensionless hydraulic head change in the wellbore (s_w/H_0)
S_{si}	specific storage in region i [L^{-1}]
t	time [T]
t_D	dimensionless time ($K_{r2}t/S_{s2}r_w^2$)
z	vertical co-ordinate [L]
z_D	dimensionless vertical co-ordinate
z_1	lower z co-ordinate of well screen [L]
z_{D1}	dimensionless lower z co-ordinate of well screen (z_1/z)
z_2	upper z co-ordinate of well screen [L]
z_{D2}	dimensionless upper z co-ordinate of well screen (z_2/z)
α	radial hydraulic conductivity ratio (K_{r2}/K_{r1})
β_n	eigenvalue for finite Fourier transform ($n\pi/B_D$)
ϕ_i	vertical to radial hydraulic conductivity ratio (K_{zi}/K_{ri})
γ	specific storage ratio (S_{s1}/S_{s2})
λ	$q_1/(\alpha q_2)$
λ'	$q_1'/(\alpha q_2')$ or $\sqrt{(\gamma/\alpha)}$
ρ	partial penetration ratio (b/B)
Δ	$q_1 I_1(q_1)[K_0(q_1 r_{D1})K_1(q_2 r_{D1}) - \lambda K_0(q_2 r_{D1})K_1(q_1 r_{D1})] + q_1 K_1(q_1)[I_0(q_1 r_{D1})K_1(q_2 r_{D1}) + \lambda K_0(q_2 r_{D1})I_1(q_1 r_{D1})]$
Δ'	$q_1' I_1(q_1')[K_0(q_1' r_{D1})K_1(q_2' r_{D1}) - \lambda' K_0(q_2' r_{D1})K_1(q_1' r_{D1})] + q_1' K_1(q_1')[I_0(q_1' r_{D1})K_1(q_2' r_{D1}) + \lambda K_0(q_2' r_{D1})I_1(q_1' r_{D1})]$

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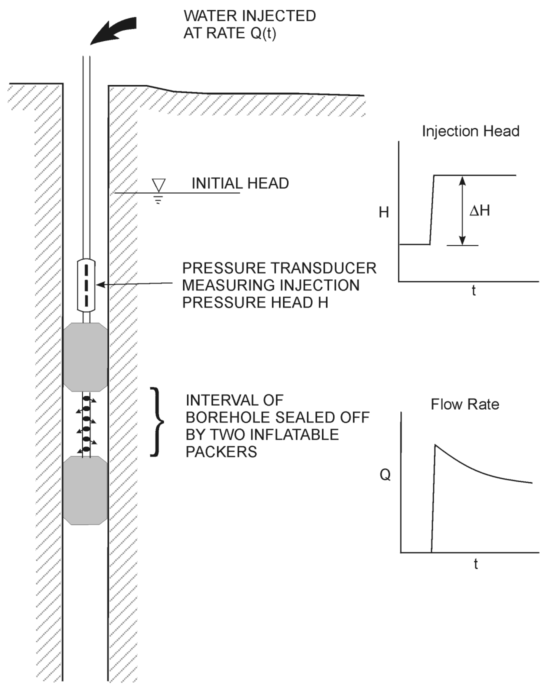

Figure 1. Schematic of constant-head test

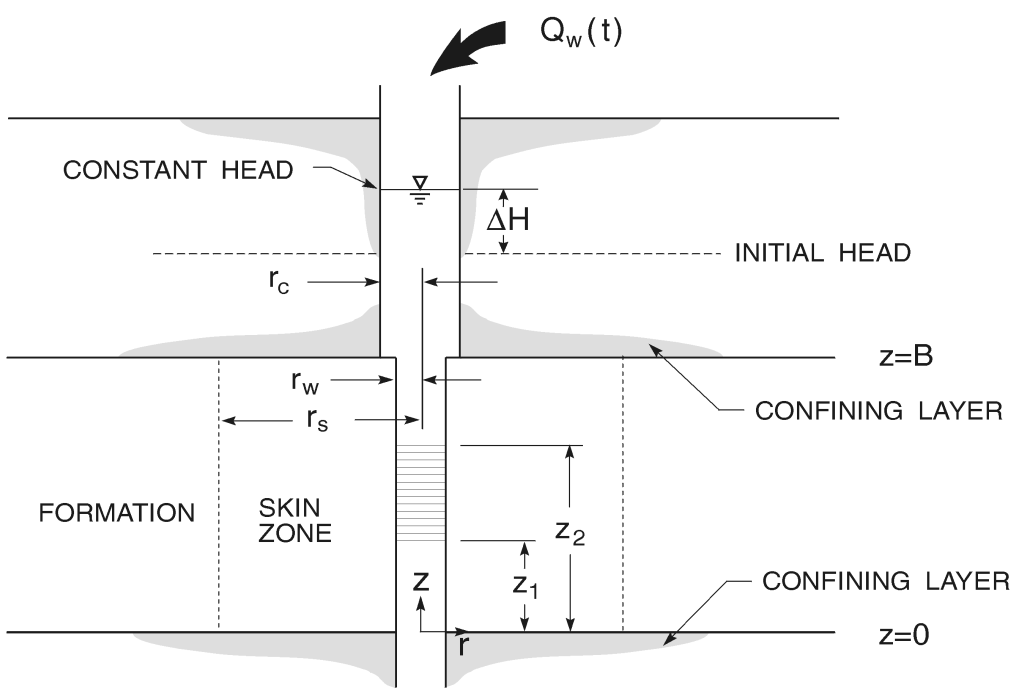

Figure 2. Definition sketch for rigorous analysis

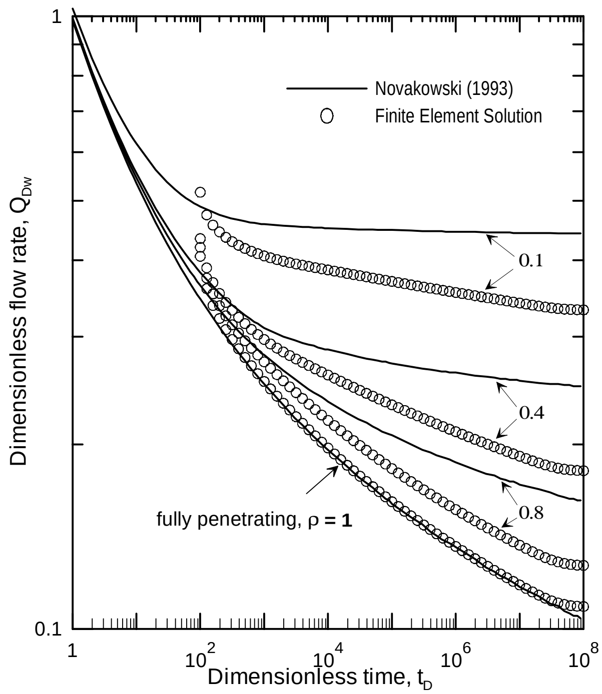

Figure 3. Comparison between Novakowski (1993) and FEM solutions for ρ = 1, 0.8, 0.4, and 0.1

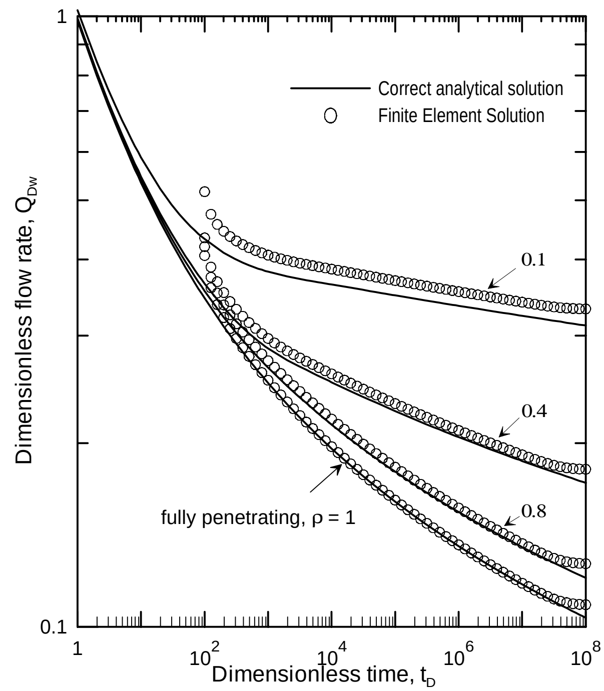

Figure 4. Comparison between correct and FEM solution for ρ = 1, 0.8, 0.4, 0.1

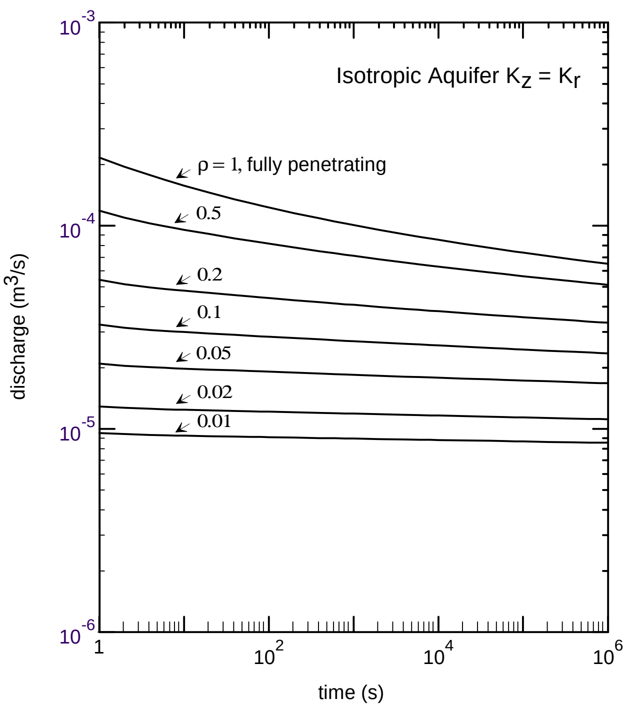

Figure 5. "Perfect" data for problem of a partially penetrating well with ρ = 1,0.5, 0.2, 0.1, 0.05, 0.02, 0.01, and K_Z=K_r

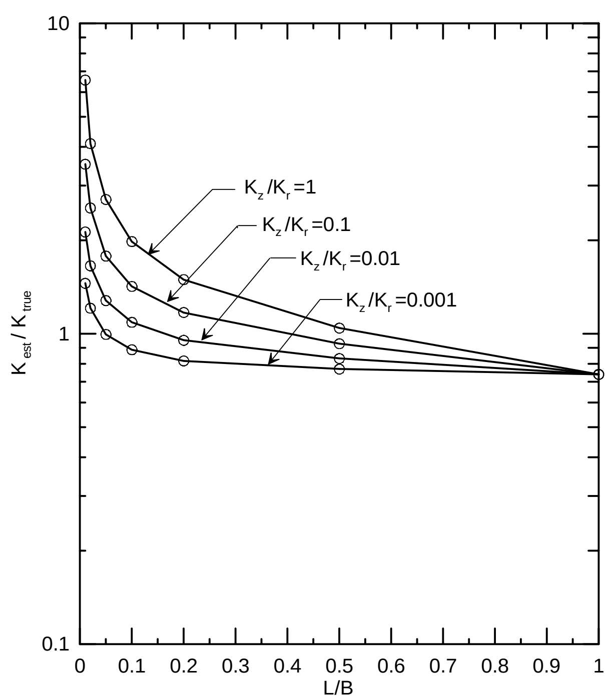

Figure 6. Conventional interpretation of data from a partially penetrating well